RULED SUBMANIFOLDS AS SELF-SIMILAR SOLUTIONS OF THE MEAN CURVATURE FLOW IN $\mathbb{R}^N$

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Abstract. We prove that the only n-dimensional ruled submanifolds of Euclidean space $\mathbb{R}^N$, with $N \geq n+1$, and in arbitrary dimension and codimension, that are non-stationary self-shrinkers, self-expanders, or translating solitons of the mean curvature flow are cylinders over self-similar plane curves. We further show that the only conical solitons of these three types are the (stationary) minimal cones. This extends to arbitrary codimension results previously known only for surfaces in $\mathbb{R}^3$.

1. Introduction

An n-dimensional submanifold Σ^n of $\mathbb{R}^N$, $N \geq n+1$, is a homothetic solution of the mean curvature flow if there exists $\lambda \in \mathbb{R}$ such that

$$\mathbf{H} = \lambda X^{\perp}, \tag{1.1}$$

where $\mathbf{H}$ is the non-normalized mean curvature vector of Σ^n and $X^{\perp}$ denotes the part of $X \in \mathbb{R}^N$ in the normal bundle of Σ^n. If $\lambda < 0$, Σ^n is called a self-shrinker, and if $\lambda > 0$, Σ^n is called a self-expander. In turn, a submanifold Σ^n of $\mathbb{R}^N$ is a translating soliton of the mean curvature flow with velocity vector $V \in \mathbb{R}^N$ if

$$\mathbf{H} = V^{\perp}. \tag{1.2}$$

Self-similar solutions to the mean curvature flow (namely self-shrinkers, self-expanders, and translating solitons) have been extensively studied in recent decades. Due to the vastness of the literature, we refer the reader to the surveys [8], [12], [13], [22], [25], [35] and [40] for an overview of the fundamental results. While examples of solitons in high codimension remain scarce, the interest in their study has grown in recent years, as shown by the following references [3], [9], [10], [11], [19], [20], [21], [29], [30], [32], [33], and [39]. Aiming to contribute to this classification, in this paper we address the case of ruled submanifolds.

An n-dimensional submanifold Σ^n of $\mathbb{R}^N$ is called a ruled submanifold if it is foliated by $(n-1)$-dimensional totally geodesic submanifolds of $\mathbb{R}^N$. Each leaf of the foliation is called a ruling of Σ^n. Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^N$ be an integral curve of the field of normal directions to the foliation of Σ^n which we will assume to be parametrized by arc length. Let $\{e_1(s), e_2(s), \ldots, e_{n-1}(s)\}$ be a set of orthonormal vector fields along γ that span the ruling of Σ^n through $\gamma(s)$. By Lemma 2.2, p. 532 of [7], we can always assume that

$$\langle e_i'(s), e_j(s)\rangle = 0, \ \forall i, j \in \{1, \ldots, n-1\}.$$

Moreover, our choice implies that

$$\gamma'(s) \perp e_i(s), \ \forall s \in I, \ \forall i \in \{1, \ldots, n-1\}.$$

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So we can give a parametrization of Σ^n by $X : U \subset \mathbb{R}^n \to \mathbb{R}^N$, where

$$X(s, t_1, \ldots, t_{n-1}) = \gamma(s) + \sum_{j=1}^{n-1} t_j e_j(s). \tag{1.3}$$

The arc-length parametrization of γ presupposes that the directrix is regular, that is, $\gamma'(s) \neq 0$. This is equivalent to the non-degeneracy of (1.3) along γ, and, as we shall see in Remark 1.4, to the condition $\beta \not\equiv 0$, where $\beta = \gamma - \sum_{j=1}^{n-1}\langle \gamma, e_j\rangle e_j$ is the component of γ orthogonal to $\text{span}\{e_1, \ldots, e_{n-1}\}$. The degenerate case in which γ is constant on some interval gives rise to the conical submanifolds, treated separately in Theorem 1.3.

Remark 1.1. Since Σ^n is a soliton of the mean curvature flow, it is a real analytic submanifold of $\mathbb{R}^N$ by elliptic regularity. The foliation by rulings is totally geodesic and hence real analytic, as is the field of directions normal to it. Consequently, γ, being an integral curve of an analytic field parametrized by arc length, is real analytic; and the Gauss map $s \mapsto \text{span}\{e_1(s), \ldots, e_{n-1}(s)\}$ is an analytic curve in the Grassmannian. The orthonormal frame $\{e_i(s)\}$ satisfying $\langle e_i'(s), e_j(s)\rangle = 0$, provided by Lemma 2.2 of [7], solves a linear differential equation with analytic coefficients and is therefore analytic as well. Hence $\gamma, e_1, \ldots, e_{n-1}$ are real analytic vector-valued functions, and the parametrization X in (1.3) is analytic.

A ruled submanifold is cylindrical, or a cylinder over a curve, if the leaves of the foliation are parallel. This means that, in the parametrization (1.3), $e_j'(s) = 0$, for every $j = 1, \ldots, n-1$. In this case, we say that Σ^n is a cylinder over the curve $\gamma \subset \mathbb{R}^N$.

It can be proved (see Section 2) that, for the parametrization (1.3), the mean curvature vector of Σ^n satisfies

$$\mathbf{H} = |X_s|^{-2} X_{ss}^{\perp}.$$

Thus, a ruled submanifold of $\mathbb{R}^N$ is a homothetic solution of the mean curvature flow if

$$|X_s|^{-2}(X_{ss})^{\perp} = \lambda X^{\perp},$$

i.e.,

$$(X_{ss} - \lambda |X_s|^2 X)^{\perp} = 0. \tag{1.4}$$

Analogously, Σ^n is a translating soliton of the mean curvature flow in $\mathbb{R}^N$ with velocity $V \in \mathbb{R}^N$ if

$$(X_{ss} - |X_s|^2 V)^{\perp} = 0. \tag{1.5}$$

Remark 1.2. If Σ^n, parametrized by (1.3), is a cylinder over the curve $\gamma \subset \mathbb{R}^N$, i.e., each $e_j(s)$ is constant, for every $j = 1, \ldots, n-1$, identify

$$\mathbb{R}^{n-1} = \text{span}\{e_1, \ldots, e_{n-1}\} \quad \text{and} \quad \mathbb{R}^{N-n+1} = \text{span}\{e_1, \ldots, e_{n-1}\}^{\perp} \subset \mathbb{R}^N.$$

By taking $\beta(s) := \gamma(s) - \sum_{j=1}^{n-1}\langle \gamma(s), e_j\rangle e_j$, we can denote the cylinder as the product $\Sigma^n = \beta \times W_0 \subset \mathbb{R}^{N-n+1} \times \mathbb{R}^{n-1} = \mathbb{R}^N$ where $\beta \subset \mathbb{R}^{N-n+1}$ and W_0 is an open set of $\mathbb{R}^{n-1}$. Moreover, by Lemma 2.1, p.5, if $\Sigma^n = \beta \times W_0$ is a self-similar solution of the mean curvature flow (self-shrinker, self-expander, or translator) then β is a plane curve self-similar solution of the two-dimensional curve shortening flow.

Remark 1.3. The curves β described in Remark 1.2 were fully classified. The self-shrinking curves of the two-dimensional curve shortening flow were classified by Abresch and Langer in [1] and the self-expanding curves were fully described by Halldorsson in [23]. In turn, the only translating curves of the two-dimensional curve shortening flow, up to rescalings and isometries of $\mathbb{R}^2$, are pieces of the grim reaper $y = -\log(\cos x)$. In this case, the cylinder $\Sigma^n = \beta \times W_0$ is called a grim reaper cylinder.

Remark 1.4. Throughout, $\beta := \gamma - \sum_{j=1}^{n-1}\langle \gamma, e_j\rangle e_j$ denotes the component of γ that is orthogonal to the ruling $\operatorname{span}\{e_1, \dots, e_{n-1}\}$. Since $|\gamma'| = 1$ and $\gamma' \perp e_j$ hold for every j, one has $\gamma' = \beta'$ in the directions transverse to the ruling. Therefore, the parametrization (1.3) is non-degenerate along γ if and only if $\beta \not\equiv 0$. The vanishing of $\beta \equiv 0$ would force $\gamma \in \operatorname{span}\{e_1, \dots, e_{n-1}\}$, rendering (1.3) degenerate; accordingly, we assume $\beta \not\equiv 0$ in the regular (non-conical) case.

The study of ruled submanifolds in Euclidean space is a classical subject in differential geometry. Early examples include planes, cones, cylinders, and helicoids. Minimal ruled submanifolds were extensively studied during the last century, with their classification completed by Barbosa, Dajczer, and Jorge in [7] (for more details about the history of this problem, see the references therein). Subsequently, Barbosa and Delgado [6] classified ruled submanifolds in space forms whose mean curvature vector has constant length. Since then, ruled submanifolds of higher codimension have been investigated in various geometric contexts, e.g., [4], [14], [15], [16], [17], [18], [27], [28], [31], and [36].

Regarding curvature flows, all the results we found in the literature concern surfaces in $\mathbb{R}^3$. Anciaux [2] proved that cylinders are the only ruled self-shrinkers in $\mathbb{R}^3$. This was later generalized by López [34], who showed that cylinders are the only ruled surfaces in $\mathbb{R}^3$ satisfying the equation $H = \lambda\langle X, \eta\rangle + c$, where η is the unit normal vector and λ, c are constants. For ruled translating solitons in $\mathbb{R}^3$, Hieu and Hoang [24] proved that they are parts of planes or cylinders over grim reaper curves. Furthermore, ruled surfaces as self-similar solutions of the inverse mean curvature flow in both $\mathbb{R}^3$ and the Lorentz-Minkowski space $\mathbb{L}^3$, were classified in [5], [26], [37] and [38].

Remark 1.5. The proofs of the aforementioned results rely essentially on tools that are specific to the three-dimensional setting. In contrast, for submanifolds of arbitrary dimension and codimension, the normal bundle may have arbitrarily large dimension, making it necessary to handle simultaneously a large number of normal vector fields. To overcome this difficulty, inspired by the approach of [6] and [7], we replace the classical tools of $\mathbb{R}^3$ by the exterior product, which allows us to work in a coordinate-free fashion with the normal bundle of arbitrary rank. This approach, while natural from a theoretical standpoint, leads to technically more involved proofs, as reflected in the length of Section 3.

Remark 1.6. If Σ^n is minimal, i.e., $\mathbf{H} = 0$, then we say that Σ^n, as a soliton of the mean curvature flow, is stationary. This means that Σ^n does not move under the action of the flow. Since the minimal ruled submanifolds were classified by Barbosa, Dajczer, and Jorge in [7], we will address only non-stationary solitons of the mean curvature flow.

This paper is motivated by the search for new examples of mean curvature flow solitons in high codimension, a setting in which remarkably few examples are known. In this search, ruled submanifolds arise as natural candidates: they form a well-understood class of submanifolds with explicit parametrizations that exist in arbitrary dimensions and codimensions.

The first result of this paper is the following classification result for ruled submanifolds Σ^n of $\mathbb{R}^N$, $N \geq n+1$, as homothetic solutions of the mean curvature flow:

Theorem 1.1. *The only ruled submanifolds Σ^n of $\mathbb{R}^N$, $N \geq n+1$, that are non-stationary self-similar solutions of the mean curvature flow are cylinders over self-shrinking or self-expanding curves of the curve shortening flow in $\mathbb{R}^2$.*

Remark 1.7. In the particular case of surfaces in $\mathbb{R}^3$, i.e., when $n = 2$ and $N = 3$, the result of Theorem 1.1 was proved by Anciaux [2] and López [34].

We also obtain an analogous result for translating solitons of the mean curvature flow:

Theorem 1.2. *The only ruled submanifolds $X : \Sigma^n \to \mathbb{R}^N$, $N \geq n+1$, that are non-stationary translating solitons of the mean curvature flow are grim reaper cylinders.*

Remark 1.8. As a particular case of Theorem 1.2, for $n = 2$ and $N = 3$, we obtain the results proved by Hieu and Hoang in [24].

Remark 1.9. The classification Theorems 1.1 and 1.2 are essentially local: given any open set $U \subset \mathbb{R}^n$ and a piece $X(U) \subset \mathbb{R}^N$ of the parametrization (1.3) satisfying (1.1) or (1.2), the conclusions hold on $X(U)$. The proofs proceed by a case distinction that is valid on the open subintervals where each alternative holds; since γ, the frame $\{e_j\}$, and the coefficients a_k are real analytic (see Remark 1.1 and Lemma 3.3) and I is connected, each alternative either holds on a discrete set or propagates to the whole connected piece. Hence the local conclusions extend globally, and, the open set U being arbitrary, the results hold without restriction.

As noted above, the arc-length parametrization of $\gamma : I \to \mathbb{R}^N$ in (1.3) requires γ to be regular in the interval I. Thus, the case when $\gamma(s) = p_0 \in \mathbb{R}^N$ is constant in some interval J requires separate treatment. In this case, we call $\Sigma^n = X(U)$ a conical submanifold. We conclude this paper by classifying conical submanifolds of $\mathbb{R}^N$ that are self-similar solutions of the mean curvature flow.

Theorem 1.3. *Every conical self-shrinker, self-expander, or translating soliton of the mean curvature flow is minimal and hence stationary. Equivalently, the regular part of such a cone is an open subset of a minimal cone.*

The paper is organized as follows. In Section 2 we introduce some basic calculations and tools to prove the main theorems, including the conventions and the precise local framework. Section 3 is dedicated to the proof of Theorem 1.1, while Theorem 1.2 is proved in Section 4. We conclude the paper by proving Theorem 1.3 in Section 5.

2. Preliminaries

Let Σ^n be a ruled submanifold of $\mathbb{R}^N$, $N \geq n+1$, parametrized by (1.3), such that

$$\langle e_i(s), e_j(s)\rangle = \delta_{ij}, \quad \langle e_i'(s), e_j(s)\rangle = 0, \quad \text{and} \quad \langle \gamma'(s), e_i(s)\rangle = 0,$$

for every $i, j = 1, \ldots, n-1$. Since

$$X_s = \gamma'(s) + \sum_{j=1}^{n-1} t_j e_j'(s) \quad \text{and} \quad X_{t_j} = e_j(s),$$

we have

$$\begin{aligned} g_{11} &= \langle X_s, X_s\rangle = |X_s|^2, \\ g_{1j} &= \langle X_s, X_{t_j}\rangle = \langle \gamma'(s), e_j(s)\rangle + \sum_{i=1}^{n-1} t_i \langle e_i'(s), e_j(s)\rangle = 0, \\ g_{ij} &= \langle X_{t_i}, X_{t_j}\rangle = \langle e_i(s), e_j(s)\rangle = \delta_{ij}. \end{aligned}$$

Thus, the metric of X is

$$dX^2 = |X_s|^2 ds^2 + \sum_{j=1}^{n-1} dt_j^2.$$

In order to calculate the second fundamental form, let $\{\eta_1, \ldots, \eta_{N-n}\}$ be an orthonormal frame of the normal bundle of Σ^n. Using that

$$X_{ss} = \gamma''(s) + \sum_{j=1}^{n-1} t_j e_j''(s), \quad X_{st_j} = e_j'(s), \quad \text{and} \quad X_{t_i t_j} = 0,$$

we have

$$b^\alpha_{11} = \langle X_{ss}, \eta_\alpha\rangle, \quad b^\alpha_{1j} = \langle X_{st_j}, \eta_\alpha\rangle = \langle e_j', \eta_\alpha\rangle, \quad b^\alpha_{ij} = \langle X_{t_it_j}, \eta_\alpha\rangle = 0,$$

for every $\alpha \in \{1, \dots, N-n\}$. Denoting by $\mathcal{B}^\alpha = (b^\alpha_{ij})_{n\times n}$ and $\mathcal{G} = (g_{ij})_{n\times n}$, we obtain

$$\mathcal{B}^\alpha\mathcal{G}^{-1} = \begin{bmatrix} \langle X_{ss},\eta_\alpha\rangle & \langle e_1',\eta_\alpha\rangle & \cdots & \langle e_{n-1}',\eta_\alpha\rangle \\ \langle e_1',\eta_\alpha\rangle & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_{n-1}',\eta_\alpha\rangle & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} |X_s|^{-2} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} |X_s|^{-2}\langle X_{ss},\eta_\alpha\rangle & \langle e_1',\eta_\alpha\rangle & \cdots & \langle e_{n-1}',\eta_\alpha\rangle \\ |X_s|^{-2}\langle e_1',\eta_\alpha\rangle & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ |X_s|^{-2}\langle e_{n-1}',\eta_\alpha\rangle & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the mean curvature in the direction of η_α is

$$H_\alpha = \operatorname{trace}(\mathcal{B}^\alpha\mathcal{G}^{-1}) = |X_s|^{-2}\langle X_{ss}, \eta_\alpha\rangle,$$

and therefore, the mean curvature vector is

$$\mathbf{H} = \sum_{\alpha=1}^{N-n} H_\alpha\eta_\alpha = \sum_{\alpha=1}^{N-n} |X_s|^{-2}\langle X_{ss},\eta_\alpha\rangle\eta_\alpha = |X_s|^{-2}(X_{ss})^\perp.$$

A submanifold Σ^n of $\mathbb{R}^N$ is a homothetic solution of the mean curvature flow if

$$\mathbf{H} = \lambda X^\perp, \ \lambda \in \mathbb{R},$$

and it is a translating soliton of the mean curvature flow with velocity vector V if

$$\mathbf{H} = V^\perp.$$

Therefore, a ruled submanifold Σ^n of $\mathbb{R}^N$ is a homothetic solution of the mean curvature flow if

$$|X_s|^{-2}(X_{ss})^\perp = \lambda X^\perp,$$

i.e.,

$$(X_{ss} - \lambda|X_s|^2 X)^\perp = 0. \tag{2.1}$$

Analogously, Σ^n is a translating soliton of the mean curvature flow if

$$(X_{ss} - |X_s|^2 V)^\perp = 0. \tag{2.2}$$

In the next result, we give a characterization of the cylinders over curves. This fact is essentially known, but we provide a complete proof here for the sake of completeness. Here, using that in such cylinders each $e_j(s)$ is constant, we identify the subspaces

$$\mathbb{R}^{n-1} = \operatorname{span}\{e_1, \dots, e_{n-1}\} \subset \mathbb{R}^N$$

and

$$\mathbb{R}^{N-n+1} = \operatorname{span}\{e_1, \dots, e_{n-1}\}^\perp \subset \mathbb{R}^N.$$

Lemma 2.1. *Let Σ^n, parametrized by (1.3), be a cylinder over a curve $\gamma \subset \mathbb{R}^N$, where each e_j is constant. If Σ^n is a homothetic solution or a translating soliton of the mean curvature flow, then the curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^{N-n+1}$, given by*

$$\beta(s) = \gamma(s) - \sum_{j=1}^{n-1}\langle\gamma(s), e_j\rangle e_j,$$

is a plane curve self-similar solution of the two-dimensional curve shortening flow.

Proof. If Σ^n is a cylinder over the curve γ, then $e_j' = 0$ for every $j = 1, \dots, n-1$. This implies that $X_s = \gamma'$, $X_{ss} = \gamma''$ and the only non-zero principal curvature of the second fundamental form in the direction of η_α is $\langle \gamma'', \eta_\alpha \rangle$. Therefore, (2.1) is equivalent to

$$(\gamma'' - \lambda\gamma)^\perp = 0.$$

By using that $\{\gamma', e_1, \dots, e_{n-1}, \eta_1, \dots, \eta_{N-n}\}$ is an orthonormal frame of $\mathbb{R}^N$, we obtain that $\Sigma^n = \gamma \times \mathbb{R}^{n-1}$ is a homothetic solution of the mean curvature flow if γ satisfies the second order vectorial ordinary differential equation

$$\gamma'' = \lambda\gamma - \lambda\langle \gamma, \gamma' \rangle \gamma' + \sum_{j=1}^{n-1} \langle \gamma'' - \lambda\gamma, e_j \rangle e_j. \tag{2.3}$$

Now, defining

$$\beta = \gamma - \sum_{j=1}^{n-1} \langle \gamma, e_j \rangle e_j,$$

we have that $\beta \subset \mathbb{R}^{N-n+1}$, $\beta' = \gamma'$, $\beta'' = \gamma''$ and $\langle \beta', \beta \rangle = \langle \gamma', \gamma \rangle$ (here we used that $\langle \gamma', e_j \rangle = 0$). Replacing this information in (2.3), we obtain that β satisfies

$$\beta''(s) = \lambda[\beta(s) - \langle \beta'(s), \beta(s) \rangle \beta'(s)]. \tag{2.4}$$

In order to prove that β is a plane curve, let us show that $\Pi(s) := \text{span}\{\beta'(s), \beta''(s)\}$ does not depend on s. As $\beta(s) \in \Pi(s)$ by (2.4), this will imply that β is a plane curve. Let $W(s) = \beta'(s) \wedge \beta''(s)$, where $\wedge$ denotes the exterior (wedge) product of vectors in $\mathbb{R}^N$. By differentiating (2.4), there exist functions $a(s)$ and $b(s)$ such that

$$\beta'''(s) = a(s)\beta'(s) + b(s)\beta''(s).$$

This implies

$$\begin{aligned} W'(s) &= \frac{d}{ds}(\beta'(s) \wedge \beta''(s)) \\ &= \beta'(s) \wedge \beta'''(s) \\ &= b(s)\beta'(s) \wedge \beta''(s) \\ &= b(s)W(s), \end{aligned}$$

i.e, fixing the initial value $s_0 \in I$,

$$W(s) = \exp\left(\int_{s_0}^{s} b(u)du \right) W(s_0).$$

Therefore, $W(s)$ is a scalar multiple of $W(s_0)$, and thus, $\Pi(s) = \Pi(s_0) =: \Pi$ does not depend on s. Notice that in the special case when $W(s_0) = 0$, then $W(s) = 0$ for every s. This implies that $\beta'(s)$ and $\beta''(s)$ are collinear for every s and, by (2.4), β lies in the same line; i.e., β is a straight line and Σ^n is an affine linear subspace and hence minimal.

Moreover, since $|\beta'(s)| = |\gamma'(s)| = 1$, s is also the arc length parameter for β. Thus, $\beta''(s)$ is the curvature vector of β and $\beta(s) - \langle \beta(s), \beta'(s) \rangle \beta'(s)$ is the projection of $\beta(s)$ in its normal direction on the plane Π. Therefore, (2.4) is exactly the equation of the self-shrinkers (for $\lambda < 0$) and self-expanders (for $\lambda > 0$) of the two-dimensional curve shortening flow.

Analogously, (2.2) is equivalent to

$$(\gamma'' - V)^\perp = 0,$$

which implies that $\Sigma^n = \gamma \times \mathbb{R}^{n-1}$ is a translating soliton of the mean curvature flow if γ solves the ODE

$$\gamma'' = V - \langle V, \gamma' \rangle \gamma' + \sum_{j=1}^{n-1} \langle \gamma'' - V, e_j \rangle e_j. \tag{2.5}$$

Defining $\beta \subset \mathbb{R}^{N-n+1}$ as before and $V_0 = V - \sum_{j=1}^{n-1} \langle V, e_j \rangle e_j \in \mathbb{R}^{N-n+1}$, we have that (2.5) is equivalent to

$$\beta''(s) = V_0 - \langle V_0, \beta'(s) \rangle \beta'(s). \tag{2.6}$$

By using an argument entirely analogous to the case of homothetic solutions, defining $\Pi(s) = \operatorname{span}\{\beta'(s), \beta''(s)\}$ and $W(s) = \beta'(s) \wedge \beta''(s)$, we obtain that

$$W(s) = \exp\left(- \int_{s_0}^{s} \langle V_0, \beta'(u) \rangle du \right) W(s_0),$$

and thus, $\Pi(s) = \Pi(s_0) =: \Pi$. By defining $\alpha(s) := \beta(s) - \beta(s_0)$, as $\alpha'(s) = \beta'(s) \in \Pi$, we have $\alpha(s) \in \Pi$, i.e., β lies in the two-dimensional affine linear subspace $\beta(s_0) + \Pi \subset \mathbb{R}^{N-n+1}$. Moreover, as $\beta''(s)$ is the curvature vector of β and $V_0 - \langle V_0, \beta'(s) \rangle \beta'(s)$ is the projection of V_0 in the normal direction of β in Π, Equation (2.6) is the equation of the translating solitons of the two-dimensional curve shortening flow. □

The following Gram-type identity will be used twice, in Sections 3 and 5, to detect minimality.

Lemma 2.2. *For a ruled submanifold Σ^n parametrized by (1.3) under the conditions above, denoting by $\langle\langle \cdot, \cdot \rangle\rangle$ the inner product of the exterior algebra and by $E = e_1 \wedge \cdots \wedge e_{n-1}$, one has*

$$\langle\langle E \wedge X_s \wedge X_{ss}, E \wedge X_s \wedge X_{ss} \rangle\rangle = |X_s|^6 \, |\mathbf{H}|^2. \tag{2.7}$$

In particular, $E \wedge X_s \wedge X_{ss} = 0$ if and only if $\mathbf{H} = 0$, i.e., Σ^n is minimal.

Proof. Since $\{e_1, \ldots, e_{n-1}\}$ is orthonormal and $\langle e_i, X_s \rangle = 0$ for all i (because $X_s = \gamma' + \sum_{j=1}^{n-1} t_j e_j'$ and $\langle e_i, \gamma' \rangle = \langle e_i, e_j' \rangle = 0$), the Gram matrix determinant of the family $\{e_1, \ldots, e_{n-1}, X_s, X_{ss}\}$ equals

$$\begin{aligned}
&\langle\langle E \wedge X_s \wedge X_{ss}, E \wedge X_s \wedge X_{ss} \rangle\rangle \\
&= \det \left[\begin{array}{ccc|c|c}
 & & & \langle e_1, X_s \rangle & \langle e_1, X_{ss} \rangle \\
 & I_{n-1} & & \vdots & \vdots \\
 & & & \langle e_{n-1}, X_s \rangle & \langle e_{n-1}, X_{ss} \rangle \\
\hline
\langle X_s, e_1 \rangle & \cdots & \langle X_s, e_{n-1} \rangle & |X_s|^2 & \langle X_s, X_{ss} \rangle \\
\hline
\langle X_{ss}, e_1 \rangle & \cdots & \langle X_{ss}, e_{n-1} \rangle & \langle X_{ss}, X_s \rangle & |X_{ss}|^2
\end{array}\right] \\
&= \det \left[\begin{array}{ccc|c|c}
 & & & 0 & \langle e_1, X_{ss} \rangle \\
 & I_{n-1} & & \vdots & \vdots \\
 & & & 0 & \langle e_{n-1}, X_{ss} \rangle \\
\hline
0 & \cdots & 0 & |X_s|^2 & \langle X_s, X_{ss} \rangle \\
\hline
\langle X_{ss}, e_1 \rangle & \cdots & \langle X_{ss}, e_{n-1} \rangle & \langle X_{ss}, X_s \rangle & |X_{ss}|^2
\end{array}\right].
\end{aligned}$$

Expanding this determinant gives

$$\begin{aligned}
&\langle\langle E \wedge X_s \wedge X_{ss}, E \wedge X_s \wedge X_{ss} \rangle\rangle \\
&\quad = |X_s|^2 \left(|X_{ss}|^2 - \sum_{j=1}^{n-1} \langle X_{ss}, e_j \rangle^2 - \left\langle X_{ss}, \tfrac{X_s}{|X_s|} \right\rangle^2 \right) \\
&\quad = |X_s|^2 \, |X_{ss}^{\perp}|^2.
\end{aligned}$$

Since $\mathbf{H} = |X_s|^{-2} X_{ss}^{\perp}$, we have $|X_{ss}^{\perp}|^2 = |X_s|^4 |\mathbf{H}|^2$, and (2.7) follows. □

We conclude this section with the following elementary lemmas:

Lemma 2.3. *Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be arbitrary functions and $Z_f = \{x \in U; f(x) = 0\}$ and $Z_g = \{x \in U; g(x) = 0\}$. If $fg \equiv 0$ on U and $Z_f \subseteq Z_g$, then $g \equiv 0$ on U.*

Proof. Let $x \in U$. Since $fg \equiv 0$, we have $f(x)g(x) = 0$, so $f(x) = 0$ or $g(x) = 0$; that is, $x \in Z_f \cup Z_g$. Hence $U = Z_f \cup Z_g$. As $Z_f \subseteq Z_g$, this gives $U = Z_g$, so $g \equiv 0$ on U. □

Lemma 2.4. *Let $U \subseteq \mathbb{R}^n$ be a connected open set, and let f, g be real analytic functions on U satisfying $fg \equiv 0$. Then at least one of f or g vanishes identically.*

Proof. The ring of real analytic functions on a connected open set is an integral domain. Indeed, suppose $f \not\equiv 0$. By the identity theorem, the zero set $Z(f)$ has empty interior, so its complement $U \setminus Z(f)$ is open and dense. Since $fg \equiv 0$, the function g vanishes on this dense set, and by continuity $g \equiv 0$ on U. Thus $f \equiv 0$ or $g \equiv 0$. □

3. Homothetic solutions

This section is devoted to the proof of Theorem 1.1. In view of its length, we will divide it into four lemmas and a conclusion. We start with the following lemma.

Lemma 3.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^N$ be a smooth curve parametrized by the arc length and $\{e_1(s), \dots, e_{n-1}(s)\}$ an orthonormal frame along $\gamma(s)$ such that $\langle e_i(s), e_j'(s)\rangle = 0$ and $\langle \gamma'(s), e_j(s)\rangle = 0$, for every $i, j = 1, \dots, n-1$. A ruled submanifold $X : \Sigma^n \to \mathbb{R}^N$, parametrized by*

$$X(s, t_1, \dots, t_{n-1}) = \gamma(s) + \sum_{k=1}^{n-1} t_k e_k(s),$$

is a homothetic solution of the mean curvature flow if and only if the curves $\gamma(s)$ and $e_j(s)$, $j = 1, \dots, n-1$, satisfy the system of equations

$$\begin{cases} E \wedge \gamma' \wedge \gamma'' = \lambda E \wedge \gamma' \wedge \gamma; \\ E \wedge \gamma' \wedge e_j'' + E \wedge e_j' \wedge \gamma'' = \lambda E \wedge e_j' \wedge \gamma + 2\lambda \langle \gamma', e_j'\rangle E \wedge \gamma' \wedge \gamma; \\ E \wedge e_i' \wedge e_j'' + E \wedge e_j' \wedge e_i'' = 2\lambda \langle \gamma', e_i'\rangle E \wedge e_j' \wedge \gamma + 2\lambda \langle \gamma', e_j'\rangle E \wedge e_i' \wedge \gamma \\ \qquad\qquad + 2\lambda \langle e_i', e_j'\rangle E \wedge \gamma' \wedge \gamma; \\ 0 = \lambda |\omega|^2 E \wedge \omega \wedge \gamma, \end{cases} \tag{3.1}$$

for every $i, j, k \in \{1, \dots, n-1\}$. Here $\wedge$ denotes the exterior product of vectors in $\mathbb{R}^N$,

$$\omega(t_1, \dots, t_{n-1}, s) = \sum_{j=1}^{n-1} t_j e_j'(s), \quad \text{and} \quad E(s) = e_1(s) \wedge e_2(s) \wedge \cdots \wedge e_{n-1}(s).$$

Proof. The equation $(X_{ss} - \lambda |X_s|^2 X)^{\perp} = 0$ means that

$$X_{ss} - \lambda |X_s|^2 X \in T\Sigma^n = \operatorname{span}\{X_{t_1}, X_{t_2}, \dots, X_{t_{n-1}}, X_s\}.$$

This implies

$$X_{t_1} \wedge X_{t_2} \wedge \cdots \wedge X_{t_{n-1}} \wedge X_s \wedge (X_{ss} - \lambda |X_s|^2 X) = 0. \tag{3.2}$$

Since $X_{t_i} = e_i$, Equation (3.2) is equivalent to

$$E \wedge X_s \wedge X_{ss} = \lambda |X_s|^2 E \wedge X_s \wedge X. \tag{3.3}$$

Expanding both sides of (3.3) and using that

$$E \wedge \gamma' \wedge e_j = 0 = E \wedge e_j' \wedge e_i, \tag{3.4}$$

we obtain

$$\begin{aligned} E \wedge X_s \wedge X_{ss} &= E \wedge \left(\gamma' + \sum_{j=1}^{n-1} t_j e_j'\right) \wedge \left(\gamma'' + \sum_{j=1}^{n-1} t_j e_j''\right) \\ &= E \wedge \gamma' \wedge \gamma'' + \sum_{j=1}^{n-1} t_j (E \wedge \gamma' \wedge e_j'' + E \wedge e_j' \wedge \gamma'') \\ &\quad + \sum_{i,j=1}^{n-1} t_i t_j E \wedge e_i' \wedge e_j'' \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \lambda |X_s|^2 E \wedge X_s \wedge X &= \lambda \left(|\gamma'|^2 + 2\sum_{j=1}^{n-1} t_j \langle \gamma', e_j' \rangle + \sum_{i,j=1}^{n-1} t_i t_j \langle e_i', e_j' \rangle\right) \\ &\quad \times E \wedge \left(\gamma' + \sum_{j=1}^{n-1} t_j e_j'\right) \wedge \left(\gamma + \sum_{j=1}^{n-1} t_j e_j\right) \\ &= \lambda \left(|\gamma'|^2 + 2\sum_{j=1}^{n-1} t_j \langle \gamma', e_j' \rangle + \sum_{i,j=1}^{n-1} t_i t_j \langle e_i', e_j' \rangle\right) \\ &\quad \times \left(E \wedge \gamma' \wedge \gamma + \sum_{j=1}^{n-1} t_j E \wedge e_j' \wedge \gamma\right) \\ &= \lambda |\gamma'|^2 E \wedge \gamma' \wedge \gamma \\ &\quad + \lambda \sum_{j=1}^{n-1} t_j \left(|\gamma'|^2 E \wedge e_j' \wedge \gamma + 2\langle \gamma', e_j' \rangle E \wedge \gamma' \wedge \gamma\right) \\ &\quad + \lambda \sum_{i,j=1}^{n-1} t_i t_j (2\langle \gamma', e_i' \rangle E \wedge e_j' \wedge \gamma + \langle e_i', e_j' \rangle E \wedge \gamma' \wedge \gamma) \\ &\quad + \lambda \sum_{i,j,k=1}^{n-1} t_i t_j t_k \langle e_i', e_j' \rangle E \wedge e_k' \wedge \gamma. \end{aligned}$$

Substituting these calculations in (3.3), using the equality of polynomials in $(t_1, \ldots, t_{n-1})$ and that $|\gamma'| = 1$ (since γ is parametrized by arc length), we obtain the three first equations (3.1), by comparing the coefficients of degree zero, one, and two. Comparing the coefficients of degree three and using symmetrization we obtain

$$0 = \lambda \langle e_i', e_j' \rangle E \wedge e_k' \wedge \gamma + \lambda \langle e_i', e_k' \rangle E \wedge e_j' \wedge \gamma + \lambda \langle e_j', e_k' \rangle E \wedge e_i' \wedge \gamma.$$

Multiplying by $t_i t_j t_k$ and summing, we have

$$0 = 3\lambda \sum_{i,j,k=1}^{n-1} t_i t_j t_k \langle e_i', e_j' \rangle E \wedge e_k' \wedge \gamma = 3\lambda \sum_{i,j,k=1}^{n-1} \langle t_i e_i', t_j e_j' \rangle E \wedge (t_k e_k') \wedge \gamma = 3\lambda |\omega|^2 E \wedge \omega \wedge \gamma.$$

This gives the fourth equation and concludes the proof. □

As for $\lambda = 0$ we have the minimal submanifolds classified by Barbosa, Dajczer, and Jorge in [7], from now on we will consider only the cases when $\lambda \neq 0$.

Lemma 3.2. *If Σ^n is a ruled submanifold of $\mathbb{R}^N$ parametrized by (1.3), then, under the conditions of Lemma 3.1 and assuming that $\lambda \neq 0$, for every $k = 1, \ldots, n-1$, there exist functions $a_k : I \to \mathbb{R}$ such that*

$$e_k' = a_k \beta, \quad \text{where} \quad \beta := \gamma - \sum_{j=1}^{n-1} \langle \gamma, e_j \rangle e_j,$$

and the following system of equations holds:

$$\begin{cases} E \wedge \gamma' \wedge (\gamma'' - \lambda\gamma) = 0; \\ \left(a_j' - 2\lambda a_j \langle \gamma', \gamma \rangle - a_j \sum_{p=1}^{n-1} a_p \langle \gamma, e_p \rangle \right) E \wedge \gamma' \wedge \gamma + a_j E \wedge \gamma \wedge \gamma'' = 0; \\ a_i a_j \left(1 + \lambda |\beta|^2\right) E \wedge \gamma \wedge \gamma' = 0. \end{cases} \tag{3.6}$$

Proof. From the last equation of the system (3.1), p.8 and assuming that $\lambda \neq 0$, we have

$$|\omega|^2 E \wedge \omega \wedge \gamma = 0$$

globally on Σ^n. Since $f = |\omega|^4$ and $g = |E \wedge \omega \wedge \gamma|^2$ are real analytic functions, using Lemma 2.3 we obtain

$$E \wedge \omega \wedge \gamma = 0.$$

globally on Σ^n. Since $E \wedge \omega \wedge \gamma = \sum_{k=1}^{n-1} t_k E \wedge e_k' \wedge \gamma$ is a polynomial of degree one in $t_1, \ldots, t_{n-1}$, we obtain that $E \wedge e_k' \wedge \gamma = 0$ for every $k = 1, \ldots, n-1$ globally on Σ^n. This means that $e_k' \in \operatorname{span}\{\gamma, e_1, \ldots, e_{n-1}\}$, i.e.,

$$e_k' = a_k \gamma + \sum_{p=1}^{n-1} b_p^{(k)} e_p,$$

for some functions $a_k(s)$ and $b_p^{(k)}(s)$. As $\langle e_k', e_j \rangle = 0$, we obtain $b_j^{(k)} = -a_k \langle \gamma, e_j \rangle$. Thus

$$e_k' = a_k \left(\gamma - \sum_{p=1}^{n-1} \langle \gamma, e_p \rangle e_p \right) = a_k \beta. \tag{3.7}$$

Differentiating (3.7) and using that $\langle \gamma', e_p \rangle = 0$ for every p, we obtain

$$e_k'' = a_k' \left(\gamma - \sum_{p=1}^{n-1} \langle \gamma, e_p \rangle e_p \right) + a_k \left(\gamma' - \sum_{p=1}^{n-1} \langle \gamma, e_p' \rangle e_p - \sum_{p=1}^{n-1} \langle \gamma, e_p \rangle e_p' \right). \tag{3.8}$$

Now, we will replace (3.7) and (3.8) in the second and third equations of the system (3.1). For the second equation, using (3.4) we compute

$$\begin{aligned}
&E\wedge\gamma'\wedge e_j''+E\wedge e_j'\wedge\gamma''\\
&\qquad=E\wedge\gamma'\wedge\left[a_j'\left(\gamma-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p\right)\right]\\
&\qquad\quad+E\wedge\gamma'\wedge\left[a_j\left(\gamma'-\sum_{p=1}^{n-1}\langle\gamma,e_p'\rangle e_p-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p'\right)\right]\\
&\qquad\quad+E\wedge\left[a_j\left(\gamma-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p\right)\right]\wedge\gamma''\\
&\qquad=a_j'E\wedge\gamma'\wedge\gamma-a_j\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle E\wedge\gamma'\wedge e_p'+a_jE\wedge\gamma\wedge\gamma''\\
&\qquad=a_j'E\wedge\gamma'\wedge\gamma-a_j\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle E\wedge\gamma'\wedge\left[a_p\left(\gamma-\sum_{q=1}^{n-1}\langle\gamma,e_q\rangle e_q\right)\right]\\
&\qquad\quad+a_jE\wedge\gamma\wedge\gamma''\\
&\qquad=\left[a_j'-a_j\sum_{p=1}^{n-1}a_p\langle\gamma,e_p\rangle\right]E\wedge\gamma'\wedge\gamma+a_jE\wedge\gamma\wedge\gamma''
\end{aligned}$$

and

$$\begin{aligned}
&\lambda E\wedge e_j'\wedge\gamma+2\lambda\langle\gamma',e_j'\rangle E\wedge\gamma'\wedge\gamma\\
&\qquad\qquad=2\lambda\left\langle\gamma',a_j\left(\gamma-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p\right)\right\rangle E\wedge\gamma'\wedge\gamma\\
&\qquad\qquad=2\lambda a_j\langle\gamma',\gamma\rangle E\wedge\gamma'\wedge\gamma.
\end{aligned}$$

So the second equation of (3.1) becomes

$$\left(a_j'-2\lambda a_j\langle\gamma',\gamma\rangle-a_j\sum_{p=1}^{n-1}a_p\langle\gamma,e_p\rangle\right)E\wedge\gamma'\wedge\gamma+a_jE\wedge\gamma\wedge\gamma''=0. \tag{3.9}$$

Here, we used that

$$0=E\wedge\gamma'\wedge\gamma'=E\wedge\gamma'\wedge e_p=E\wedge e_p\wedge\gamma''$$

by the properties of the exterior product and that $E \wedge e_j' \wedge \gamma = 0$ by hypothesis. For the third equation of (3.1), using (3.7) and (3.8), each term of the left hand side is

$$\begin{aligned}
&E \wedge e_i' \wedge e_j'' \\
&= a_i a_j' E \wedge \left(\gamma - \sum_{p=1}^{n-1} \langle \gamma, e_p\rangle e_p\right) \wedge \left(\gamma - \sum_{p=1}^{n-1} \langle \gamma, e_p\rangle e_p\right) \\
&\quad + a_i a_j E \wedge \left(\gamma - \sum_{p=1}^{n-1} \langle \gamma, e_p\rangle e_p\right) \wedge \left(\gamma' - \sum_{p=1}^{n-1} \langle \gamma, e_p'\rangle e_p - \sum_{p=1}^{n-1} \langle \gamma, e_p\rangle e_p'\right) \\
&= a_i a_j \left(E \wedge \gamma \wedge \gamma' - \sum_{p=1}^{n-1} \langle \gamma, e_p\rangle E \wedge \gamma \wedge e_p'\right) \\
&= a_i a_j E \wedge \gamma \wedge \gamma'.
\end{aligned}$$

In turn, since $E \wedge e_k' \wedge \gamma = 0$, the right hand side becomes

$$2\lambda \langle e_i', e_j'\rangle E \wedge \gamma' \wedge \gamma = 2\lambda a_i a_j |\beta|^2 E \wedge \gamma' \wedge \gamma.$$

Thus, the third equation of the system (3.1) becomes

$$a_i a_j \left(1 + \lambda |\beta|^2\right) E \wedge \gamma \wedge \gamma' = 0. \tag{3.10}$$

This concludes the proof. □

Lemma 3.3. *Let Σ^n be a ruled submanifold of $\mathbb{R}^N$ parametrized by (1.3) and satisfying Lemma 3.1 and Lemma 3.2. Then, one of the following holds:*

(i) Σ^n is a cylinder over a curve;
(ii) Σ^n is a minimal submanifold;
(iii) $\lambda < 0$, the a_j are constants for every $j = 1, \dots, n-1$, and there exist a smooth curve $F : I \subset \mathbb{R} \to \mathbb{R}^N$, constant vectors $u_1, \dots, u_{n-1}$ and constants $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that $1 + \lambda |F'(s)|^2 = 0$,

$$e_j(s) = a_j F(s) + u_j \quad \text{and} \quad \gamma(s) = F'(s) + \sum_{j=1}^{n-1} \left(\frac{a_j}{-\lambda} s + c_j\right) e_j(s).$$

Proof. Since $\gamma, e_1, \dots, e_{n-1}$ are real analytic, so are $E = e_1 \wedge \cdots \wedge e_{n-1}$, the derivatives e_k', and $\beta = \gamma - \sum_{j=1}^{n-1} \langle \gamma, e_j\rangle e_j$. Taking the inner product of (3.7) with β gives

$$a_k(s) = \frac{\langle e_k'(s), \beta(s)\rangle}{\langle \beta(s), \beta(s)\rangle}.$$

Since β is real analytic and I is connected, either $\beta \equiv 0$ or the zero set $Z = \{s \in I : \beta(s) = 0\}$ is discrete. The case $\beta \equiv 0$ cannot occur, since β is the component of γ orthogonal to $\operatorname{span}\{e_1, \dots, e_{n-1}\}$, and its vanishing would render the parametrization (1.3) degenerate. Otherwise, on $I \setminus Z$ the function

$$a_k = \frac{\langle e_k', \beta\rangle}{\langle \beta, \beta\rangle}$$

is real analytic. By (3.7), e_k' is collinear with β, so $|a_k| = |e_k'|/|\beta|$ remains locally bounded near each point of Z; hence a_k extends real analytically across Z by removable singularity. Therefore each a_k is real analytic on all of I.

By taking $i=j$ in the last equation of (3.6) and squaring the norm, we have

$$a_i^4\left(1+\lambda|\beta|^2\right)^4|E\wedge\gamma\wedge\gamma'|^2=0. \tag{3.11}$$

Since (3.11) is a product of real analytic functions, by Lemma 2.4, for $f=a_i^4$ and $g=\left(1+\lambda|\beta|^2\right)^2|E\wedge\gamma\wedge\gamma'|^2$, at least one of f or g vanishes identically. If $g\not\equiv 0$, then $a_i\equiv 0$ everywhere on Σ^n, i.e., the e_i are constants and Σ^n is globally a cylinder over γ. Therefore, we will assume that $g\equiv 0$, i.e.,

$$\left(1+\lambda|\beta|^2\right)^2|E\wedge\gamma\wedge\gamma'|^2=0 \tag{3.12}$$

everywhere on Σ^n. Applying Lemma 2.4 again, we obtain that

$$1+\lambda|\beta|^2=0 \quad \text{or} \quad E\wedge\gamma\wedge\gamma'=0$$

everywhere on Σ^n.

First case: $E\wedge\gamma\wedge\gamma'=0$. The first and second equations of (3.6) give $E\wedge\gamma\wedge\gamma''=0$ and $E\wedge\gamma'\wedge\gamma''=0$. Hence, from the second and third equations of (3.1) (remember we are assuming that $E\wedge e_j'\wedge\gamma=0$),

$$E\wedge\gamma'\wedge e_j''+E\wedge e_j'\wedge\gamma''=0 \quad \text{and} \quad E\wedge e_i'\wedge e_j''+E\wedge e_j'\wedge e_i''=0. \tag{3.13}$$

Replacing (3.13) and $E\wedge\gamma'\wedge\gamma''=0$ in (3.5), we obtain $E\wedge X_s\wedge X_{ss}=0$. By the Gram identity (2.7) of Lemma 2.2, this gives $|X_s|^6|\mathbf{H}|^2=0$, so $\mathbf{H}=0$ and Σ^n is minimal and stationary.

Second case: $1+\lambda|\beta|^2=0$. This implies that $\lambda<0$ (i.e., Σ^n is a self-shrinker) and $|\beta|^2=-1/\lambda$ is constant, i.e., $\langle\beta',\beta\rangle=0$. Since

$$\beta'=\gamma'-\sum_{p=1}^{n-1}\langle\gamma,e_p'\rangle e_p-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p',$$

we have

$$\begin{aligned}
\langle\beta',\beta\rangle&=\left\langle\gamma'-\sum_{p=1}^{n-1}\langle\gamma,e_p'\rangle e_p-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p',\gamma-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle e_p\right\rangle\\
&=\langle\gamma',\gamma\rangle-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle\langle\gamma,e_p'\rangle\\
&=\langle\gamma',\gamma\rangle-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle\left\langle\gamma,a_p\left(\gamma-\sum_{q=1}^{n-1}\langle\gamma,e_q\rangle e_q\right)\right\rangle\\
&=\langle\gamma',\gamma\rangle-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle a_p\left[|\gamma|^2-\sum_{q=1}^{n-1}\langle\gamma,e_q\rangle^2\right]\\
&=\langle\gamma',\gamma\rangle-\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle a_p|\beta|^2\\
&=\langle\gamma',\gamma\rangle+(1/\lambda)\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle a_p.
\end{aligned}$$

This gives

$$\sum_{p=1}^{n-1}\langle\gamma,e_p\rangle a_p=-\lambda\langle\gamma',\gamma\rangle. \tag{3.14}$$

Replacing (3.14) in the second equation of the system (3.6), we obtain

$$\left(a_j' - \lambda a_j \langle \gamma', \gamma \rangle\right) E \wedge \gamma' \wedge \gamma + a_j E \wedge \gamma \wedge \gamma'' = 0. \tag{3.15}$$

On the other hand, by the first equation of (3.6) and the fact that $e_1, \ldots, e_{n-1}$ and γ' are orthonormal, we have $\gamma'' - \lambda\gamma \in \operatorname{span}\{e_1, \ldots, e_{n-1}, \gamma'\}$, i.e.,

$$\gamma'' = \lambda\gamma + c\gamma' + \sum_{p=1}^{n-1} d_p e_p$$

for some functions $c(s), d_1(s), \ldots, d_{n-1}(s)$. Since

$$0 = \langle \gamma'', \gamma' \rangle = \lambda \langle \gamma, \gamma' \rangle + c\langle \gamma', \gamma' \rangle + \sum_{p=1}^{n-1} d_p \langle e_p, \gamma' \rangle = \lambda \langle \gamma', \gamma \rangle + c,$$

we get $c = -\lambda\langle \gamma, \gamma' \rangle$. This implies

$$\begin{aligned} E \wedge \gamma \wedge \gamma'' &= E \wedge \gamma \wedge \left(\lambda\gamma + c\gamma' + \sum_{p=1}^{n-1} d_p e_p \right) \\ &= E \wedge \gamma \wedge (c\gamma') \\ &= cE \wedge \gamma \wedge \gamma' \\ &= -\lambda\langle \gamma, \gamma' \rangle E \wedge \gamma \wedge \gamma' \\ &= \lambda\langle \gamma, \gamma' \rangle E \wedge \gamma' \wedge \gamma. \end{aligned}$$

Replacing this expression in (3.15), we obtain

$$a_j' E \wedge \gamma' \wedge \gamma = 0. \tag{3.16}$$

Assuming that $E \wedge \gamma' \wedge \gamma \neq 0$ (we dealt with the case $E \wedge \gamma' \wedge \gamma = 0$ earlier), we obtain that the a_j are constant. As $e_j' = a_j \beta$, denoting by $F(s) = \int_0^s \beta(\xi) d\xi$, there exist constant vectors $u_1, u_2, \ldots u_{n-1}$ such that

$$e_j(s) = a_j F(s) + u_j, \quad \forall j \in \{1, \ldots, n-1\}.$$

Now, by

$$\gamma = \beta + \sum_{k=1}^{n-1} \langle \gamma, e_k \rangle e_k, \tag{3.17}$$

we have

$$\begin{aligned} \gamma' &= \beta' + \sum_{k=1}^{n-1} (\langle \gamma, e_k \rangle)' e_k + \sum_{k=1}^{n-1} \langle \gamma, e_k \rangle e_k' \\ &= \beta' + \sum_{k=1}^{n-1} (\langle \gamma, e_k \rangle)' e_k + \left(\sum_{k=1}^{n-1} a_k \langle \gamma, e_k \rangle \right) \beta. \end{aligned} \tag{3.18}$$

Assuming $a_j \neq 0$, $e_j' = a_j \beta$ gives the equivalence of $\langle e_i, e_j' \rangle = 0$ and $\langle e_i, \beta \rangle = 0$. Thus, from (3.18), we have

$$0 = \langle \gamma', e_j \rangle = \langle \beta', e_j \rangle + (\langle \gamma, e_j \rangle)'.$$

By using that $e_j(s) = a_j F(s) + u_j$ and $F''(s) = \beta'(s)$, we obtain

$$\frac{d}{ds} \langle \gamma, e_j \rangle = -\langle \beta', e_j \rangle = -\langle F'', a_j F + u_j \rangle = -a_j \langle F'', F \rangle - \langle F'', u_j \rangle. \tag{3.19}$$

On the other hand, since

$$0 = \langle e_j, e_j' \rangle = \langle a_j F + u_j, a_j F' \rangle = a_j (a_j \langle F, F' \rangle + \langle u_j, F \rangle),$$

we have, after differentiation, that

$$\begin{aligned} 0 &= a_j \langle F', F' \rangle + a_j \langle F, F'' \rangle + \langle u_j, F'' \rangle \\ &= a_j |\beta|^2 + a_j \langle F, F'' \rangle + \langle u_j, F'' \rangle \\ &= -\frac{a_j}{\lambda} + a_j \langle F, F'' \rangle + \langle u_j, F'' \rangle. \end{aligned} \tag{3.20}$$

Replacing (3.20) in (3.19) we obtain

$$\frac{d}{ds} \langle \gamma, e_j \rangle = -\frac{a_j}{\lambda},$$

i.e.,

$$\langle \gamma(s), e_j(s) \rangle = -\frac{a_j}{\lambda} s + c_j, \quad c_j = \text{constant}. \tag{3.21}$$

On the other hand, if $a_j = 0$, then $e_j(s)$ is constant. This gives that $\frac{d}{ds}\langle \gamma, e_j \rangle = \langle \gamma', e_j \rangle = 0$ and (3.21) still holds. Thus, since $\beta(s) = F'(s)$, we obtain by (3.17) that

$$\gamma(s) = F'(s) + \sum_{k=1}^{n-1} \left(\frac{a_k}{-\lambda} s + c_k \right) (a_k F(s) + u_k). \tag{3.22}$$

□

The next lemma is a technical tool to help us address case (iii) of Lemma 3.3.

Lemma 3.4. *Let $F : I \subset \mathbb{R} \to \mathbb{R}^N$ be a smooth curve such that $1 + \lambda |F'(s)|^2 = 0$, where $\lambda < 0$. Given $a_1, \ldots, a_{n-1}, c_1, \ldots, c_{n-1} \in \mathbb{R}$, and $u_1, \ldots, u_{n-1} \in \mathbb{R}^N$ constant vectors, define*

$$e_k(s) = a_k F(s) + u_k \quad \text{and} \quad \gamma(s) = F'(s) + \sum_{k=1}^{n-1} \left(\frac{a_k}{-\lambda} s + c_k \right) e_k(s). \tag{3.23}$$

If

$$X(s, t_1, \ldots, t_{n-1}) = \gamma(s) + \sum_{k=1}^{n-1} t_k e_k(s)$$

defines a ruled self-shrinker of the mean curvature flow such that

$$|\gamma'(s)| = 1, \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad \langle e_i, e_j' \rangle = 0, \quad \text{and} \quad \langle \gamma', e_j \rangle = 0,$$

then, writing $m_k(s) = -\frac{a_k}{\lambda} s + c_k$, it holds

$$\begin{cases} E \wedge F'' \wedge F''' + \left(\sum_{k=1}^{n-1} a_k m_k \right) E \wedge F' \wedge F''' = \left[\lambda + \left(\sum_{k=1}^{n-1} a_k m_k \right)^2 + 2 \sum_{k=1}^{n-1} \frac{a_k^2}{\lambda} \right] E \wedge F'' \wedge F'; \\ a_j E \wedge F' \wedge F''' = 0; \\ |F''|^2 = 1 + \frac{1}{\lambda} \left(\sum_{k=1}^{n-1} a_k m_k \right)^2 + \sum_{k=1}^{n-1} \frac{a_k^2}{\lambda^2}. \end{cases} \tag{3.24}$$

Proof. By (3.23), we have

$$\begin{cases} \gamma = F' + \sum_{k=1}^{n-1} m_k e_k; \\ \gamma' = F'' + \sum_{k=1}^{n-1} \left(-\frac{a_k}{\lambda}\right) e_k + \left(\sum_{k=1}^{n-1} a_k m_k\right) F'; \\ \gamma'' = F''' + 2\left(\sum_{k=1}^{n-1} -\frac{a_k^2}{\lambda}\right) F' + \left(\sum_{k=1}^{n-1} a_k m_k\right) F''. \end{cases} \tag{3.25}$$

Since $\langle F', F'\rangle = -1/\lambda$ implies $\langle F'', F'\rangle = 0$, and

$$\begin{aligned} 0 = \langle e_k', e_k\rangle &= a_k\langle F', e_k\rangle \\ &\overset{d/ds}{\Longrightarrow} a_k\langle F'', e_k\rangle + a_k\langle F', e_k'\rangle = 0 \\ &\Longrightarrow\ a_k\langle F'', e_k\rangle = -a_k^2|F'|^2 = a_k^2/\lambda, \end{aligned}$$

we have, by the arc length condition, that

$$\begin{aligned} 1 = \langle\gamma', \gamma'\rangle &= |F''|^2 + \sum_{k=1}^{n-1}\frac{a_k^2}{\lambda^2} + \left(\sum_{k=1}^{n-1} a_k m_k\right)^2 |F'|^2 + 2\sum_{k=1}^{n-1}\frac{a_k}{-\lambda}\langle F'', e_k\rangle \\ &= |F''|^2 - \sum_{k=1}^{n-1}\frac{a_k^2}{\lambda^2} + \frac{1}{-\lambda}\left(\sum_{k=1}^{n-1} a_k m_k\right)^2. \end{aligned}$$

This gives the third equation of the system (3.24).

In order to verify the second equation of the system (3.24), we will replace the expressions for γ, γ' and γ'' given by (3.25) in the second equation of the system (3.1). Indeed, we have

$$\begin{aligned} E\wedge\gamma'\wedge e_j'' + E\wedge e_j'\wedge\gamma'' &= E\wedge\left(F'' + \sum_{k=1}^{n-1}\left(-\frac{a_k}{\lambda}\right)e_k + \left(\sum_{k=1}^{n-1} a_k m_k\right)F'\right)\wedge(a_j F'') \\ &\quad + E\wedge(a_j F')\wedge\left(F''' + 2\left(\sum_{k=1}^{n-1} -\frac{a_k^2}{\lambda}\right)F' + \left(\sum_{k=1}^{n-1} a_k m_k\right)F''\right) \\ &= 2a_j\left(\sum_{k=1}^{n-1} a_k m_k\right)E\wedge F'\wedge F'' + a_j E\wedge F'\wedge F''' \end{aligned}$$

and

$$\begin{aligned} &2\lambda\langle\gamma', e_j'\rangle E\wedge\gamma'\wedge\gamma \\ &= 2\lambda\left\langle F'' + \sum_{k=1}^{n-1}\left(-\frac{a_k}{\lambda}\right)e_k + \left(\sum_{k=1}^{n-1} a_k m_k\right)F', a_j F'\right\rangle \\ &\quad\times E\wedge\left(F'' + \sum_{k=1}^{n-1}\left(-\frac{a_k}{\lambda}\right)e_k + \left(\sum_{k=1}^{n-1} a_k m_k\right)F'\right)\wedge\left(F' + \sum_{k=1}^{n-1} m_k e_k\right) \\ &= 2\lambda a_j|F'|^2\left(\sum_{k=1}^{n-1} a_k m_k\right)E\wedge F''\wedge F'. \end{aligned}$$

Thus, the second equation of system (3.1) is equivalent to

$$2a_j\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F'\wedge F''+a_jE\wedge F'\wedge F'''$$
$$=2\lambda a_j|F'|^2\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F''\wedge F',$$

i.e.,

$$2a_j(1+\lambda|F'|^2)\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F'\wedge F''+a_jE\wedge F'\wedge F'''=0.$$

Using that $1+\lambda|F'|^2=0$, the second equation of the system (3.1) is equivalent to

$$a_jE\wedge F'\wedge F'''=0. \tag{3.26}$$

Now, let us verify the equivalent conditions for the first equation of the system (3.1). Since

$$\begin{aligned}E\wedge\gamma'\wedge\gamma''&=E\wedge\left(F''+\sum_{k=1}^{n-1}\left(-\frac{a_k}{\lambda}\right)e_k+\left(\sum_{k=1}^{n-1}a_km_k\right)F'\right)\\&\quad\wedge\left(F'''+2\left(\sum_{k=1}^{n-1}-\frac{a_k^2}{\lambda}\right)F'+\left(\sum_{k=1}^{n-1}a_km_k\right)F''\right)\\&=E\wedge F''\wedge F'''+2\left(\sum_{k=1}^{n-1}-\frac{a_k^2}{\lambda}\right)E\wedge F''\wedge F'\\&\quad+\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F'\wedge F'''+\left(\sum_{k=1}^{n-1}a_km_k\right)^2E\wedge F'\wedge F''\\&=E\wedge F''\wedge F'''+\left[\left(\sum_{k=1}^{n-1}a_km_k\right)^2+2\sum_{k=1}^{n-1}\frac{a_k^2}{\lambda}\right]E\wedge F'\wedge F''\\&\quad+\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F'\wedge F'''\end{aligned}$$

and $E\wedge\gamma'\wedge\gamma=E\wedge F''\wedge F'$, the first equation of (3.1) is equivalent to

$$E\wedge F''\wedge F'''+\left[\left(\sum_{k=1}^{n-1}a_km_k\right)^2+2\sum_{k=1}^{n-1}\frac{a_k^2}{\lambda}\right]E\wedge F'\wedge F''$$
$$+\left(\sum_{k=1}^{n-1}a_km_k\right)E\wedge F'\wedge F'''=\lambda E\wedge F''\wedge F'.$$

Using (3.26) we obtain

$$E\wedge F''\wedge F'''=\left[\lambda+\left(\sum_{k=1}^{n-1}a_km_k\right)^2+2\sum_{k=1}^{n-1}\frac{a_k^2}{\lambda}\right]E\wedge F''\wedge F'. \tag{3.27}$$

This completes the proof of the equations of the system (3.24).

□

Now, we conclude the proof of Theorem 1.1.

Conclusion of the proof of Theorem 1.1. We will use Lemma 3.4 to prove that the class of submanifolds described in the item (iii) of Lemma 3.3 contains only cylinder over curves. By the Lemma 3.4, such submanifolds satisfy the system of equations (3.24). By its second equation and by Lemma 2.4, $a_i = 0$ or $E \wedge F' \wedge F''' = 0$ everywhere on Σ^n. If $E \wedge F' \wedge F''' \neq 0$, then $a_i = 0$ for every $i = 1, \ldots, n-1$, i.e., Σ^n is a cylinder over a curve. Thus, let us assume that

$$E \wedge F' \wedge F''' = 0,$$

everywhere on Σ^n. This is equivalent to

$$F'''(s) = A(s)F'(s) + \sum_{j=1}^{n-1} h_j(s)e_j(s),$$

for some functions $A : I \to \mathbb{R}$ and $h_j : I \to \mathbb{R}$, $j = 1, \ldots, n-1$. Replacing this expression in the first equation of (3.24) gives

$$A(s) = \lambda + \left(\sum_{k=1}^{n-1} a_k m_k(s)\right)^2 + 2\sum_{k=1}^{n-1} \frac{a_k^2}{\lambda} \tag{3.28}$$

or

$$E \wedge F'' \wedge F = 0, \tag{3.29}$$

everywhere on Σ^n by Lemma 2.4 again. Since $\langle F', e_j\rangle = \langle \beta, e_j\rangle = 0$, we have $h_j = \langle F''', e_j\rangle$. But differentiating successively, we obtain

$$\begin{aligned}
\langle F', e_j\rangle = 0 &\overbrace{\Longrightarrow}^{d/ds} \langle F'', e_j\rangle + \langle F', e_j'\rangle = 0\\
&\Longrightarrow \langle F'', e_j\rangle + a_j|F'|^2 = 0\\
&\Longrightarrow \langle F'', e_j\rangle = a_j/\lambda\\
&\overbrace{\Longrightarrow}^{d/ds} \langle F''', e_j\rangle + \langle F'', e_j'\rangle = 0\\
&\Longrightarrow \langle F''', e_j\rangle + a_j\langle F'', F'\rangle = 0\\
&\Longrightarrow \langle F''', e_j\rangle = 0,
\end{aligned} \tag{3.30}$$

i.e., $h_j \equiv 0$. Thus,

$$F'''(s) = A(s)F'(s).$$

Assuming (3.28) and differentiating $\langle F', F'\rangle = -1/\lambda$ twice, we obtain

$$\begin{aligned}
\langle F', F'\rangle = -1/\lambda &\overbrace{\Longrightarrow}^{d/ds} \langle F'', F'\rangle = 0\\
&\overbrace{\Longrightarrow}^{d/ds} \langle F''', F'\rangle + \langle F'', F''\rangle = 0\\
&\Longrightarrow (-1/\lambda)A(s) + |F''(s)|^2 = 0.
\end{aligned} \tag{3.31}$$

By replacing the expression of the third equation of the system (3.24) in the last equation of (3.31) we have

$$\frac{1}{\lambda^2}\sum_{k=1}^{n-1} a_k^2 = 0,$$

i.e., all the $a_j = 0$, $j = 1, \ldots, n-1$, which is a contradiction with the assumption that some $a_j \neq 0$.

In turn, assuming (3.29), we get

$$F''(s) = B(s)F'(s) + \sum_{j=1}^{n-1} g_j(s)e_j(s), \tag{3.32}$$

for some functions $B : I \to \mathbb{R}$ and $g_j : I \to \mathbb{R}$, $j = 1, \dots, n-1$. Since $\langle F', e_j\rangle = 0$ and $\langle F'', F'\rangle = 0$, we obtain, by taking the inner product of (3.32) with F', that $B(s) \equiv 0$. On the other hand, by (3.30), $a_j/\lambda = \langle F'', e_j\rangle = g_j$, which gives

$$F''(s) = \frac{1}{\lambda}\sum_{j=1}^{n-1} a_j e_j(s).$$

Taking the norm and comparing with the third equation of (3.24) we get

$$\begin{aligned} 0 = 1 + \frac{1}{\lambda}\left(\sum_{j=1}^{n-1} a_j m_j(s)\right)^2 &\implies \left(\sum_{j=1}^{n-1}\frac{a_j^2}{-\lambda}\right)s + \sum_{j=1}^{n-1} a_j c_j = \pm\sqrt{-\lambda} \\ &\implies \sum_{j=1}^{n-1}\frac{a_j^2}{-\lambda} = 0, \end{aligned}$$

which implies that $a_j = 0$ for every $j = 1, \dots, n-1$. This contradiction shows us that the class of submanifolds described in the item (iii) of Lemma 3.3 contain only cylinder over curves. Therefore, if Σ^n is non-stationary (i.e., Σ^n is not a minimal submanifold), then it is a cylinder over a curve. By Lemma 2.1 and Remark 1.3, the profile curve is a self-shrinking (Abresch–Langer) or self-expanding (Halldorsson) curve of the curve shortening flow, which completes the proof. □

4. Translating solitons

The classification of ruled submanifolds as translating solitons of the mean curvature flow follows the same lines as the classification of homothetic solutions, but the analysis is simpler. Consequently, we omit some technical details to avoid redundancy.

Proof of Theorem 1.2. First, note that $(X_{ss} - |X_s|^2 V)^\perp = 0$ is equivalent to

$$E \wedge X_s \wedge X_{ss} = |X_s|^2 E \wedge X_s \wedge V,$$

where, as before, $E = e_1 \wedge \cdots \wedge e_{n-1}$. Proceeding step by step, as in the proof of Lemma 3.1, we obtain the system

$$\begin{cases} E \wedge \gamma' \wedge \gamma'' = E \wedge \gamma' \wedge V; \\ E \wedge \gamma' \wedge e_j'' + E \wedge e_j' \wedge \gamma'' = E \wedge e_j' \wedge V + 2\langle \gamma', e_j'\rangle E \wedge \gamma' \wedge V; \\ E \wedge e_i' \wedge e_j'' + E \wedge e_j' \wedge e_i'' = 2\langle \gamma', e_i'\rangle E \wedge e_j' \wedge V + 2\langle \gamma', e_j'\rangle E \wedge e_i' \wedge V \\ \qquad\qquad + 2\langle e_i', e_j'\rangle E \wedge \gamma' \wedge V; \\ 0 = |\omega|^2 E \wedge \omega \wedge V, \end{cases} \tag{4.1}$$

where $\omega = \sum_{j=1}^{n-1} t_j e_j'$ as before. By the same analysis as in the proof of Lemma 3.1, using Lemma 2.3, the fourth equation implies that $E \wedge \omega \wedge V = 0$ everywhere on Σ^n. Since it is a polynomial

of degree one in $t_1, \dots, t_{n-1}$ we obtain that $E \wedge e_k' \wedge V = 0$ for every k everywhere on Σ^n. Then there exist functions $a_k(s)$ and $b_1^{(k)}(s), \dots, b_{n-1}^{(k)}(s)$ such that

$$e_k' = a_k V + \sum_{p=1}^{n-1} b_p^{(k)} e_p.$$

By using that $\langle e_k', e_j \rangle = 0$, we obtain $b_j^{(k)} = -a_k \langle V, e_j \rangle$. Thus

$$e_k' = a_k \left(V - \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p \right) =: a_k V_0.$$

Now we analyze the third equation of (4.1). Since

$$e_k'' = a_k' \left(V - \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p \right) - a_k \left(\sum_{p=1}^{n-1} \langle V, e_p' \rangle e_p + \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p' \right), \tag{4.2}$$

we have

$$\begin{aligned} E \wedge e_i' \wedge e_j'' &= a_i a_j' E \wedge \left(V - \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p \right) \wedge \left(V - \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p \right) \\ &\quad - a_i a_j E \wedge \left(V - \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p \right) \wedge \left(\sum_{p=1}^{n-1} \langle V, e_p' \rangle e_p + \sum_{p=1}^{n-1} \langle V, e_p \rangle e_p' \right) \\ &= -a_i a_j \sum_{p=1}^{n-1} \langle V, e_p \rangle E \wedge V \wedge e_p' \\ &= 0. \end{aligned}$$

Here the last equality follows from the collinearity $e_p' = a_p V_0$ established above: since $V_0 = V - \sum_{q=1}^{n-1} \langle V, e_q \rangle e_q$ differs from V by a combination of the e_q, we have $E \wedge V \wedge e_p' = a_p E \wedge V \wedge V_0 = a_p E \wedge V \wedge V = 0$. This is precisely the term that, in the homothetic case, survived as $a_i a_j (1 + \lambda |\beta|^2) E \wedge \gamma \wedge \gamma'$ and led to the analysis of case (iii) of Lemma 3.3; its vanishing here is what makes the translating case simpler. Thus the third equation of the system (4.1) is equivalent to

$$\langle e_i', e_j' \rangle E \wedge \gamma' \wedge V = 0.$$

By taking $i = j$ we obtain

$$|e_i'|^2 E \wedge \gamma' \wedge V = 0. \tag{4.3}$$

Since all the functions in (4.3) are real analytic, by Lemma 2.4, we have that $e_i' \equiv 0$ or $E \wedge \gamma' \wedge V \equiv 0$. If $E \wedge \gamma' \wedge V \not\equiv 0$, then $e_i' \equiv 0$ for every $i = 1, \dots, n-1$ everywhere on Σ^n and it is a cylinder over a curve. Therefore, let us consider that $E \wedge \gamma' \wedge V = 0$ everywhere on Σ^n. Replacing this information and the fact that $E \wedge e_j' \wedge V = 0$ in the system (4.1), we obtain that $E \wedge \gamma' \wedge \gamma'' = 0$ and $E \wedge \gamma' \wedge e_j'' + E \wedge e_j' \wedge \gamma'' = 0$. In turn, these facts imply that $E \wedge X_s \wedge X_{ss} = 0$ and since $|E \wedge X_s \wedge X_{ss}| = |X_s|^3 |\mathbf{H}|$, by Lemma 2.2, we conclude that $\mathbf{H} = 0$ and Σ^n is minimal, i.e., stationary.

Therefore, in all non-stationary cases, Σ^n is a cylinder over a curve; by Lemma 2.1 and Remark 1.3, the profile curve is a translating curve of the two-dimensional curve shortening flow, i.e., a piece of the grim reaper. Thus Σ^n is a grim reaper cylinder, which concludes the proof. □

5. Conical submanifolds

This section is dedicated to the proof of Theorem 1.3 on the classification of conical submanifolds of $\mathbb{R}^N$ that are self-shrinkers, self-expanders, or translating solitons of the mean curvature flow.

Proof of Theorem 1.3. The conical submanifolds are parametrized by

$$X(s, t_1, \dots, t_{n-1}) = p_0 + \sum_{j=1}^{n-1} t_j e_j(s),$$

where $p_0 \in \mathbb{R}^N$ is a constant vector.

If Σ^n is a homothetic solution, then, proceeding as in the proof of Lemma 3.1, we obtain the system

$$\begin{cases} E \wedge e_i' \wedge e_j'' + E \wedge e_j' \wedge e_i'' = 0; \\ |\omega|^2 E \wedge \omega \wedge p_0 = 0, \end{cases} \tag{5.1}$$

for $\omega = \sum_{j=1}^{n-1} t_j e_j'$ as before. On the other hand, as

$$\begin{aligned} E \wedge X_s \wedge X_{ss} &= \sum_{i,j=1}^{n-1} t_i t_j E \wedge e_i' \wedge e_j'' \\ &= \sum_{i<j} t_i t_j (E \wedge e_i' \wedge e_j'' + E \wedge e_j' \wedge e_i'') + \sum_{i=1}^{n-1} t_i^2 E \wedge e_i' \wedge e_i'' = 0 \end{aligned}$$

by the first equation of (5.1) and $|E \wedge X_s \wedge X_{ss}| = |X_s|^3 |\mathbf{H}|$ by Lemma 2.2, we conclude that Σ^n is minimal. In turn, if Σ^n is a translating soliton, then it holds

$$\begin{cases} E \wedge e_i' \wedge e_j'' + E \wedge e_j' \wedge e_i'' = 0; \\ |\omega|^2 E \wedge \omega \wedge V = 0, \end{cases}$$

with an identical analysis. □

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