

# A HOPF THEOREM UNDER A CAUCHY–RIEMANN INEQUALITY IN $E(\kappa, \tau)$ , AND ITS SHARPNESS IN EVERY GENUS

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**ABSTRACT.** Let  $\Sigma$  be a smooth, connected, oriented surface homeomorphic to the sphere, immersed in the homogeneous Riemannian 3-manifold  $E(\kappa, \tau)$ , and let  $\mathcal{Q}_{\text{AR}}$  be its original Abresch–Rosenberg differential. If the mean curvature  $H$  satisfies the Cauchy–Riemann inequality  $|dH| \leq h|\mathcal{Q}_{\text{AR}}|$  for some continuous function  $h \geq 0$ , then  $\mathcal{Q}_{\text{AR}} \equiv 0$  and  $H$  is constant; it suffices that  $h \in L^p_{\text{loc}}$  for some  $p > 2$ . For  $\tau = 0$  this is due to Alencar, do Carmo and Tribuzy (2007), and for  $\tau \neq 0$ ,  $\kappa \neq 4\tau^2$ , to Alencar, do Carmo, Fernández and Tribuzy (2007). We give a complete proof, based on the Bers–Vekua similarity principle and the Poincaré–Hopf index formula for line fields, which treats all values of  $\kappa$  at once, including the round sphere  $\kappa = 4\tau^2$ . The same argument shows that, on a closed surface of genus  $g$  satisfying the inequality, either  $\mathcal{Q}_{\text{AR}} \equiv 0$  or its zeros have total order  $4g - 4$ . The genus-zero hypothesis cannot be replaced by any other genus. Indeed, the inequality constrains  $H$  only near the zeros of  $\mathcal{Q}_{\text{AR}}$ , and a small deformation supported away from them turns any closed CMC surface with  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  into a non-CMC surface that satisfies the inequality with  $h$  constant and arbitrarily small. Using the Daniel correspondence, we show that  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  on every closed CMC surface of positive genus with  $\kappa + 4H^2 > 0$ ; hence the round sphere and every Berger sphere contain embedded non-CMC examples of every genus  $g \geq 1$ . In every Berger sphere, the Hopf torus over a simple closed curve with nonconstant geodesic curvature is an explicit non-CMC example of genus one.

## 1. INTRODUCTION

The classical Hopf differential of an immersed surface in  $\mathbb{R}^3$  is the  $(2, 0)$ -component of its second fundamental form; Hopf showed that it is holomorphic if and only if the mean curvature  $H$  is constant, and deduced that every immersed constant mean curvature (CMC) sphere in  $\mathbb{R}^3$  is round [16]. Let  $E(\kappa, \tau)$  denote the simply connected homogeneous 3-manifold that fibers over the space form  $M^2(\kappa)$  with geodesic fibers and bundle curvature  $\tau$ ; for  $\tau = 0$  this is the product  $M^2(\kappa) \times \mathbb{R}$  (the formalism is recalled in §2), and let  $\mathcal{Q}_{\text{AR}}$  be the original Abresch–Rosenberg differential [1, 2]. In the companion paper [6] we prove that, for  $\tau \neq 0$  and with no completeness or analyticity hypothesis, holomorphy of  $\mathcal{Q}_{\text{AR}}$  is *equivalent* to  $H$  being constant. Here we return to a different, classical question: how much of the holomorphy hypothesis is actually needed for rigidity when the surface is a topological sphere?

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Alencar–do Carmo–Tribuzy [5, Theorem 1.1] showed that, for spheres in the products  $M^2(c) \times \mathbb{R}$ , holomorphy can be replaced by a Cauchy–Riemann inequality, and Alencar–do Carmo–Fernández–Tribuzy [4, Theorem 1] extended this to the spaces  $E(\kappa, \tau)$  with four-dimensional isometry group and  $\tau \neq 0$ , that is,  $\kappa \neq 4\tau^2$ , for the same differential  $\mathcal{Q}_{\text{AR}}$ . We use the norm  $|\mathcal{Q}_{\text{AR}}|_g = |\mathcal{Q}_{\text{AR}}|/\lambda$  with respect to the induced metric  $g = \lambda|d\zeta|^2$ . Since the hypothesis below involves an arbitrary continuous function  $h$ , it is unchanged if  $\mathcal{Q}_{\text{AR}}$  is multiplied by a continuous nowhere vanishing function; hence it depends neither on the normalization of the differential nor on that of its norm. In particular, it is exactly the hypothesis of [4], and, for  $\tau \neq 0$  (so that  $H + i\tau$  never vanishes), it is equivalent to the corresponding hypothesis for the Fernández–Mira normalization  $\mathcal{Q}_{\text{AR}}/(H + i\tau)$ . This invariance concerns the inequality only: multiplication by a nowhere vanishing function that is not holomorphic does not, in general, preserve holomorphy.

**Theorem 1.1.** *Let  $\Sigma$  be a smooth, connected, oriented surface homeomorphic to  $S^2$ , and let  $\psi : \Sigma \rightarrow E(\kappa, \tau)$ ,  $\kappa, \tau \in \mathbb{R}$ , be an immersion satisfying*

$$|dH|_g \leq h |\mathcal{Q}_{\text{AR}}|_g, \quad h \in C^0(\Sigma, [0, \infty)).$$

*Then  $\mathcal{Q}_{\text{AR}} \equiv 0$  and  $H$  is constant. The same holds if  $h \geq 0$  is only measurable, with  $h \in L^p_{\text{loc}}(\Sigma)$  for some  $p > 2$ , and the inequality holds almost everywhere.*

**Remark 1.2.** For  $\tau = 0$ , Theorem 1.1 with continuous  $h$  is the first conclusion of [5, Theorem 1.1], and for  $\tau \neq 0$ ,  $\kappa \neq 4\tau^2$ , it is [4, Theorem 1]. In the round case  $\kappa = 4\tau^2 \neq 0$  one has  $\mathcal{Q}_{\text{AR}} = 2(H + i\tau)p$  with  $|H + i\tau| \geq |\tau| > 0$ , so the hypothesis is equivalent to  $|dH|_g \leq \tilde{h} |p d\zeta^2|_g$  for a continuous  $\tilde{h} \geq 0$ . This is a Cauchy–Riemann inequality for the classical Hopf differential of a surface in the round sphere  $S^3(\tau^2)$ , and the conclusion says that  $\Sigma$  is totally umbilic. When  $h \equiv 0$  the hypothesis is simply  $dH \equiv 0$ . In general the inequality allows  $dH \neq 0$  away from the zero set of  $\mathcal{Q}_{\text{AR}}$ . On spheres, Theorem 1.1 shows a posteriori that only CMC immersions with  $\mathcal{Q}_{\text{AR}} \equiv 0$  satisfy it, whereas on surfaces of every positive genus it is satisfied by non-CMC immersions (Proposition 5.1 and Theorem 6.1).

The proof rests on a simple mechanism. The identity (2.10), valid for every immersion and due, in another normalization, to [4, Proposition 1], turns the hypothesis of Theorem 1.1 into a differential inequality  $|(Q_{\text{AR}})_{\bar{\zeta}}| \leq a |Q_{\text{AR}}|$  for the coefficient of  $\mathcal{Q}_{\text{AR}}$  itself. The Bers–Vekua similarity principle then shows that the zeros of  $Q_{\text{AR}}$  behave like those of a holomorphic function: they are isolated and of positive order. This is all that Hopf’s index argument requires, so holomorphy can be replaced by the inequality. Schematically,

$$\begin{aligned} \text{Cauchy–Riemann inequality} &\implies \text{similarity principle} \\ &\implies \text{indices of the zeros} \implies \text{topological rigidity.} \end{aligned}$$

This is the strategy of [5, 4], where the local factorization is provided by a Chern-type lemma [10]. Our contribution to Theorem 1.1 is therefore not a new mechanism, but the following: (i) a single argument for every  $E(\kappa, \tau)$ , including the round sphere  $\kappa = 4\tau^2$ , whose isometry group has dimension six and which lies outside the statement of [4] (although the argument there applies to it as well); (ii) a complete treatment of the analytic step (Lemma 4.1) and of the passage from the local factorization to a global dichotomy on  $\Sigma$  (Corollary 4.2); and (iii) independence from the local characterization theorem of the companion paper [6]. The same mechanism gives, on a closed surface

of any genus  $g$ , the count  $\sum_j m_j = 4g - 4$  for the orders of the zeros of  $\mathcal{Q}_{\text{AR}}$  whenever  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  (Corollary 4.4), exactly as for holomorphic quadratic differentials.

We conclude only that  $H$  is constant and  $\mathcal{Q}_{\text{AR}} \equiv 0$ ; in general, embeddedness does not follow. In the round sphere a CMC sphere with  $\mathcal{Q}_{\text{AR}} \equiv 0$  is totally umbilic, hence embedded; but in some Berger spheres with  $\kappa \neq 4\tau^2$ , Torralbo [24, Theorem 1(i), Corollary 1 and Fig. 2] found rotationally invariant CMC spheres that are immersed but not embedded, contrary to the embeddedness asserted in [2, Theorem 6]. The rotational invariance of CMC spheres with  $\mathcal{Q}_{\text{AR}} \equiv 0$ , stated in [4, Theorem 1], rests on the Abresch–Rosenberg classification [2], which we do not use. The argument also applies verbatim in the products  $M^2(\kappa) \times \mathbb{R}$  (Remark 4.3).

The main new point of the paper is that the genus-zero hypothesis is sharp. By Proposition 5.1, in every Berger sphere, including the round one, non-CMC Hopf tori (preimages under the submersion  $\pi : E(\kappa, \tau) \rightarrow M^2(\kappa)$  of simple closed curves with nonconstant geodesic curvature, cf. §3.1) satisfy the inequality, even with  $h$  constant. On these tori  $\mathcal{Q}_{\text{AR}}$  has no zeros, and the inequality then holds for trivial reasons (Remark 5.2). In genus  $g \geq 2$  zeros are forced (Corollary 4.4), but the inequality is still not enough: by Theorem 6.1, every closed CMC surface with  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  admits small deformations, supported away from the zeros of  $\mathcal{Q}_{\text{AR}}$ , that preserve the inequality with  $h$  constant and make  $H$  nonconstant. In the round sphere, applying this to the Clifford torus and to Lawson’s surfaces  $\xi_{g,1}$  [21], this gives embedded non-CMC examples of every genus  $g \geq 1$ . In every Berger sphere the same holds for Torralbo’s minimal surfaces [25]: by Lemma 6.3, which combines Daniel’s sister correspondence [11] with the Abresch–Rosenberg classification in the product spaces [1],  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  on every closed CMC surface of positive genus with  $\kappa + 4H^2 > 0$  (Corollary 6.4). Moreover, the constant  $h$  can be taken arbitrarily small, and the deformed surfaces coincide with the CMC surface near the zeros of  $\mathcal{Q}_{\text{AR}}$ ; so no positive smallness threshold on  $h$ , and no condition that is satisfied by the CMC surface and depends only on the immersion near the zeros of  $\mathcal{Q}_{\text{AR}}$ , can force  $H$  to be constant in positive genus (Remark 6.7). The same perturbation, applied to Wente’s tori and to Kapouleas’ surfaces, gives immersed non-CMC examples of every genus  $g \geq 1$  in  $\mathbb{R}^3$  (Remark 6.6). Thus the genus-zero hypothesis of Theorem 1.1 cannot be replaced by any other genus. Throughout, *sharpness* refers to this topological hypothesis: we do not claim that the inequality admits an optimal constant, and indeed no positive smallness threshold on  $h$  can replace the hypothesis of genus zero (the case  $h \equiv 0$  being trivial).

Section 2 recalls Daniel’s structure equations and derives from them the identity (2.10), so that the paper is self-contained. Section 3 shows that a closed surface with  $\det S < 0$  is a torus and computes the geometry of Hopf tubes, which is needed only for the examples. Section 4 proves Theorem 1.1 and Corollary 4.4, Section 5 constructs the non-CMC Hopf tori, and Section 6 gives non-CMC examples in every genus.

## 2. PRELIMINARIES

We recall Daniel’s structure equations for an immersion into  $E(\kappa, \tau)$  [11]. For  $\tau \neq 0$ ,  $E(\kappa, \tau)$  denotes the simply connected homogeneous space admitting a Riemannian submersion  $\pi : E(\kappa, \tau) \rightarrow M^2(\kappa)$  onto the simply connected space form of curvature  $\kappa$ , with geodesic fibers and unit vertical Killing field  $\xi$

satisfying  $\bar{\nabla}_X \xi = \tau X \times \xi$ ; up to isometry this is the only such space for each fixed pair  $(\kappa, \tau)$  with  $\tau \neq 0$  [11, §2]. We write  $k := \kappa - 4\tau^2$ ; for  $\tau \neq 0$  the value  $k = 0$  (i.e.,  $\kappa = 4\tau^2$ ) is the round sphere  $S^3(\tau^2)$  of constant curvature  $\tau^2$ , fibered by the Hopf fibration, while for  $\tau = 0$  it means  $\kappa = 0$ , and  $E(0, 0) = \mathbb{R}^3$ . We use  $E(\kappa, \tau)$  in this extended sense, including the constant-curvature case  $\kappa = 4\tau^2$ ; [11] and [4] reserve the notation for  $\kappa \neq 4\tau^2$ , where the isometry group has dimension four. For  $\tau = 0$  we set  $E(\kappa, 0) := M^2(\kappa) \times \mathbb{R}$ , with  $\xi = \partial_t$ ; everything in this section holds verbatim in that case.

For a smooth oriented immersion  $\psi : \Sigma \rightarrow E(\kappa, \tau)$ , fix the unit normal  $N$ , the shape operator  $S = -\bar{\nabla}N$ , the mean curvature  $H = \frac{1}{2} \operatorname{tr} S$ , and  $u = \langle N, \xi \rangle$ . Denote by  $J$  the positive rotation by  $\pi/2$  in the tangent plane, so  $JX = N \times X$ , and by  $T = \xi - uN$  the tangential part of  $\xi$ . In a conformal parameter  $\zeta$ , with  $g = \lambda |d\zeta|^2$ , let  $p = \langle S\partial_\zeta, \partial_\zeta \rangle$  and  $A = \langle \xi, \partial_\zeta \rangle$ . In Daniel's conventions [11, Corollary 3.2 and Proposition 3.3], which follow from O'Neill's formulas for a Riemannian submersion with geodesic fibers [22, Ch. 7], the structure equations are

$$(2.1) \quad K = \det S + \tau^2 + ku^2,$$

$$(2.2) \quad (\nabla_X S)Y - (\nabla_Y S)X = ku(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

$$(2.3) \quad \nabla_X T = u(SX - \tau JX),$$

$$(2.4) \quad X(u) = -\langle SX - \tau JX, T \rangle, \quad |T|^2 + u^2 = 1.$$

The original Abresch–Rosenberg differential is  $\mathcal{Q}_{\text{AR}} = Q_{\text{AR}} d\zeta^2$ ,

$$(2.5) \quad Q_{\text{AR}} = 2(H + i\tau)p - kA^2,$$

introduced in [1] for  $\tau = 0$  and, as a solution of an abstract equation, in [2] for  $\tau \neq 0$ ; the explicit expression (2.5) for  $\tau \neq 0$  is the one used in [4] and [13]. We now derive from (2.1)–(2.4) the identity (2.10) on which the proof of Theorem 1.1 rests. For  $\kappa \neq 4\tau^2$  this identity is [4, Proposition 1], written there for  $\psi = Q(Z, Z)$  with  $Z = \sqrt{2}\partial_\zeta$ , so that  $\psi = 2Q_{\text{AR}}$ ; it also appears in [6]. We include the short derivation, which covers  $\kappa = 4\tau^2$  as well, to keep the paper self-contained.

Extend  $S$ ,  $J$ , and the Levi-Civita connection  $\nabla$  of the induced metric  $\mathbb{C}$ -bilinearly to the complexified tangent bundle. Since  $J$  is the rotation by  $\pi/2$  compatible with the complex structure underlying the conformal parameter  $\zeta$ ,  $J\partial_\zeta = i\partial_\zeta$  and  $J\partial_{\bar{\zeta}} = -i\partial_{\bar{\zeta}}$ ; and, for the conformal metric  $g = \lambda |d\zeta|^2$ ,  $\langle \partial_\zeta, \partial_\zeta \rangle = 0$ ,  $\langle \partial_\zeta, \partial_{\bar{\zeta}} \rangle = \lambda/2$ , with the only nonvanishing covariant derivatives among  $\partial_\zeta, \partial_{\bar{\zeta}}$  being

$$(2.6) \quad \nabla_{\partial_\zeta} \partial_\zeta = (\log \lambda)_\zeta \partial_\zeta, \quad \nabla_{\partial_{\bar{\zeta}}} \partial_{\bar{\zeta}} = (\log \lambda)_{\bar{\zeta}} \partial_{\bar{\zeta}}, \quad \nabla_{\partial_\zeta} \partial_{\bar{\zeta}} = \nabla_{\partial_{\bar{\zeta}}} \partial_\zeta = 0,$$

the standard formulas for isothermal coordinates. Self-adjointness of  $S$ , extended bilinearly, together with  $\langle S\partial_\zeta, \partial_{\bar{\zeta}} \rangle = \frac{\lambda}{2}H$  (immediate upon expanding  $\partial_\zeta, \partial_{\bar{\zeta}}$  in a local orthonormal frame and using  $H = \frac{1}{2} \operatorname{tr} S$ ), gives

$$(2.7) \quad \langle S\partial_{\bar{\zeta}}, \partial_\zeta \rangle = \langle S\partial_\zeta, \partial_{\bar{\zeta}} \rangle = \frac{\lambda}{2}H.$$

*The equation for  $A_{\bar{\zeta}}$ .* The  $\mathbb{R}$ -linear identity (2.3),  $\nabla_X T = u(SX - \tau JX)$ , extends  $\mathbb{C}$ -bilinearly to  $X = \partial_{\bar{\zeta}}$ :  $\nabla_{\partial_{\bar{\zeta}}} T = u(S\partial_{\bar{\zeta}} + i\tau\partial_{\bar{\zeta}})$ . Since  $T$  is tangent and  $N \perp \partial_\zeta$ ,  $A = \langle \xi, \partial_\zeta \rangle = \langle T, \partial_\zeta \rangle$ ; as  $\nabla_{\partial_{\bar{\zeta}}} \partial_\zeta = 0$  by (2.6),

$$\begin{aligned} A_{\bar{\zeta}} &= \partial_{\bar{\zeta}} \langle T, \partial_\zeta \rangle = \langle \nabla_{\bar{\zeta}} T, \partial_\zeta \rangle + \langle T, \nabla_{\bar{\zeta}} \partial_\zeta \rangle \\ &= \langle \nabla_{\bar{\zeta}} T, \partial_\zeta \rangle = u(\langle S\partial_{\bar{\zeta}}, \partial_\zeta \rangle + i\tau \langle \partial_{\bar{\zeta}}, \partial_\zeta \rangle), \end{aligned}$$

which, by (2.7), gives

$$(2.8) \quad A_{\bar{\zeta}} = \frac{u\lambda}{2}(H + i\tau).$$

*The equation for  $p_{\bar{\zeta}}$ .* The Codazzi identity (2.2), extended  $\mathbb{C}$ -bilinearly to  $(X, Y) = (\partial_{\zeta}, \partial_{\bar{\zeta}})$ , reads, using  $\langle \partial_{\zeta}, T \rangle = A$ ,  $\langle \partial_{\bar{\zeta}}, T \rangle = \bar{A}$ ,

$$(\nabla_{\zeta} S) \partial_{\bar{\zeta}} - (\nabla_{\bar{\zeta}} S) \partial_{\zeta} = ku(\bar{A} \partial_{\zeta} - A \partial_{\bar{\zeta}}).$$

Pairing with  $\partial_{\zeta}$ : since  $\langle \partial_{\zeta}, \partial_{\zeta} \rangle = 0$  and  $\langle \partial_{\bar{\zeta}}, \partial_{\zeta} \rangle = \lambda/2$ , the right side becomes  $-ku\lambda A/2$ . On the left, since  $\nabla_{\bar{\zeta}} \partial_{\zeta} = 0$ ,  $(\nabla_{\bar{\zeta}} S) \partial_{\zeta} = \nabla_{\bar{\zeta}}(S \partial_{\zeta})$ , and

$$\langle (\nabla_{\bar{\zeta}} S) \partial_{\zeta}, \partial_{\zeta} \rangle = \langle \nabla_{\bar{\zeta}}(S \partial_{\zeta}), \partial_{\zeta} \rangle = \partial_{\bar{\zeta}} \langle S \partial_{\zeta}, \partial_{\zeta} \rangle - \langle S \partial_{\zeta}, \nabla_{\bar{\zeta}} \partial_{\zeta} \rangle = p_{\bar{\zeta}}.$$

Likewise, since  $\nabla_{\zeta} \partial_{\bar{\zeta}} = 0$ ,  $(\nabla_{\zeta} S) \partial_{\bar{\zeta}} = \nabla_{\zeta}(S \partial_{\bar{\zeta}})$ , and, using this time the non-vanishing term  $\nabla_{\zeta} \partial_{\zeta} = (\log \lambda)_{\zeta} \partial_{\zeta}$  of (2.6),

$$\begin{aligned} \partial_{\zeta} \langle S \partial_{\bar{\zeta}}, \partial_{\zeta} \rangle &= \langle \nabla_{\zeta}(S \partial_{\bar{\zeta}}), \partial_{\zeta} \rangle + \langle S \partial_{\bar{\zeta}}, \nabla_{\zeta} \partial_{\zeta} \rangle \\ &= \langle \nabla_{\zeta}(S \partial_{\bar{\zeta}}), \partial_{\zeta} \rangle + (\log \lambda)_{\zeta} \cdot \frac{\lambda}{2} H. \end{aligned}$$

Substituting  $\langle S \partial_{\bar{\zeta}}, \partial_{\zeta} \rangle = \lambda H/2$  from (2.7) on the left and expanding  $\partial_{\zeta}(\lambda H/2) = \frac{1}{2} \lambda_{\zeta} H + \frac{1}{2} \lambda H_{\zeta}$ , the term  $\frac{1}{2} \lambda_{\zeta} H$  cancels exactly against  $(\log \lambda)_{\zeta} \cdot \frac{\lambda}{2} H = \frac{1}{2} \lambda_{\zeta} H$ , leaving  $\langle \nabla_{\zeta}(S \partial_{\bar{\zeta}}), \partial_{\zeta} \rangle = \frac{\lambda}{2} H_{\zeta}$ . Substituting both pairings into the displayed Codazzi identity,

$$\frac{\lambda}{2} H_{\zeta} - p_{\bar{\zeta}} = -\frac{ku\lambda A}{2},$$

i.e.

$$(2.9) \quad p_{\bar{\zeta}} = \frac{\lambda}{2}(H_{\zeta} + kuA).$$

*The holomorphy identity.* Differentiating (2.5) in  $\bar{\zeta}$  ( $\tau, k$  constant) and substituting (2.8)–(2.9),

$$\begin{aligned} (Q_{\text{AR}})_{\bar{\zeta}} &= 2H_{\bar{\zeta}}p + 2(H + i\tau)p_{\bar{\zeta}} - 2kAA_{\bar{\zeta}} \\ &= 2H_{\bar{\zeta}}p + \lambda(H + i\tau)H_{\zeta} + ku\lambda A(H + i\tau) - ku\lambda A(H + i\tau), \end{aligned}$$

using  $2(H + i\tau)p_{\bar{\zeta}} = \lambda(H + i\tau)(H_{\zeta} + kuA)$  and  $2kAA_{\bar{\zeta}} = ku\lambda A(H + i\tau)$ ; the two identical terms  $ku\lambda A(H + i\tau)$  cancel exactly, leaving, for *every* smooth immersion, with no hypothesis on  $H$ :

$$(2.10) \quad (Q_{\text{AR}})_{\bar{\zeta}} = 2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\zeta}.$$

In particular,  $Q_{\text{AR}}$  is holomorphic if and only if  $2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\zeta} = 0$ , which holds identically when  $H$  is constant. All norms below are computed in the induced metric:  $|dH|_g^2 = \lambda^{-1}|dH|^2$  and  $|Q_{\text{AR}}|_g = |Q_{\text{AR}}|/\lambda$ .

### 3. A TOPOLOGICAL OBSTRUCTION, AND HOPF TUBES

The following well-known proposition is a consequence of the Poincaré–Hopf theorem for line fields. It depends only on the condition  $\det S < 0$ , not on the ambient space; we include the short proof because it identifies the topological type of the Hopf tubes below.

**Proposition 3.1** (A topological consequence of  $\det S < 0$ ). *Let  $\Sigma$  be a closed, connected, oriented surface, smoothly immersed with unit normal  $N$  in an oriented Riemannian 3-manifold, with shape operator  $S = -\bar{\nabla}N$  satisfying  $\det S < 0$  at every point. Then  $\Sigma$  is a torus.*

*Proof.* Since  $\det S < 0$ , the eigenvalues  $\mu_1 < 0 < \mu_2$  of  $S$  are distinct at every point, and they are smooth, being  $H \mp \sqrt{H^2 - \det S}$ . Hence the spectral projector  $(S - \mu_1 \text{Id})/(\mu_2 - \mu_1)$  is smooth, and its image, the  $\mu_2$ -eigenline, is a smooth line field on  $\Sigma$  without singularities. By the Poincaré–Hopf theorem for line fields [17, Part Two, Ch. III, §2.2, p. 113],  $\chi(\Sigma) = 0$ , and by the classification of closed orientable surfaces [15, Ch. 9],  $\Sigma$  is a torus.  $\square$

**3.1. Geometry and differential of Hopf tubes.** The geometric facts in Proposition 3.2 below ( $T$  parallel,  $K = 0$ ,  $\det S = -\tau^2$ ,  $2H = k_g$ ) are well known; see Pinkall [23, Lemma 1 and Proposition 1] for Hopf tori in the round sphere and Albuje–dos Santos [3] for Hopf tori in Berger spheres. We include short proofs in our notation; the new ingredient needed here is the formula for  $Q_{\text{AR}}$ .

Given a smooth, regular, embedded curve  $\gamma$  in the base  $M^2(\kappa)$ , the associated *Hopf tube* is the preimage  $\Sigma_\gamma := \pi^{-1}(\gamma) \subset E(\kappa, \tau)$ . Since  $\pi$  is a submersion,  $\Sigma_\gamma$  is an embedded surface; it is connected when  $\gamma$  is, because the fibers are connected. We orient it by the unit normal  $N$  that is the horizontal lift of a unit normal  $\tilde{N}$  of  $\gamma$  in  $M^2(\kappa)$ . (For a regular curve  $\gamma : I \rightarrow M^2(\kappa)$  with self-intersections the preimage need not be a surface; one then works with the pullback  $\tilde{\Sigma}_\gamma = \{(s, x) \in I \times E(\kappa, \tau) : \gamma(s) = \pi(x)\}$  and the immersion  $(s, x) \mapsto x$ . Everything below is local and applies to this case, but only embedded curves are needed in this paper.) A Hopf tube is locally characterized by  $u \equiv 0$ : since  $\xi$  is tangent to each fiber and each fiber lies in  $\Sigma_\gamma$ ,  $u = \langle N, \xi \rangle = 0$ ; conversely,  $u \equiv 0$  means that  $\xi$  is everywhere tangent, and its integral curves (the fibers) locally foliate the surface. If the base is  $S^2(\kappa)$  and  $\gamma$  is a smooth simple closed curve,  $\Sigma_\gamma$  is a compact embedded surface, called a Hopf torus once Proposition 3.1 identifies its topological type; see also [3] for Hopf tori in Berger spheres.

**Proposition 3.2.** *On a Hopf tube, the field  $T = \xi$  is unit and parallel, the metric is flat, and, in the parallel frame  $(E_1, E_2) = (-JT, T)$ ,*

$$S = \begin{pmatrix} 2H & \tau \\ \tau & 0 \end{pmatrix}, \quad T(H) = 0, \quad \det S = -\tau^2.$$

*In the local coordinates  $\partial_x = E_1$ ,  $\partial_y = E_2$ , and  $\zeta = x + iy$ , the original differential has coefficient*

$$Q_{\text{AR}} = H^2 + \frac{\kappa}{4}.$$

*In particular, on a connected Hopf tube its holomorphy is equivalent to  $H$  constant. If  $k_g$  is the geodesic curvature of the base curve with respect to the unit normal whose horizontal lift is  $N$ , then  $2H = k_g$ .*

*Proof.*  $T$  is unit and parallel. On  $\Sigma_\gamma$ ,  $u \equiv 0$ , so  $T = \xi - uN = \xi$ : unit, and tangent as noted. By (2.3),  $\nabla_X T = u(SX - \tau JX)$ ; since  $u \equiv 0$ ,  $\nabla_X T \equiv 0$ :  $T$  is parallel.

*Flatness, and the parallel frame.* Since  $J$  is parallel on any oriented Riemannian surface,  $E_1 := -JT$  satisfies  $\nabla_X E_1 = -J\nabla_X T = 0$ :  $E_1$  is parallel as well, and  $(E_1, E_2) = (-JT, T)$  is a global parallel orthonormal frame. A surface with a global parallel orthonormal frame is flat:  $\langle R(E_1, E_2)E_2, E_1 \rangle$  vanishes term by term (each  $\nabla E_i \equiv 0$ , and  $[E_1, E_2] = 0$  by torsion-freeness), so  $K \equiv 0$ . Being parallel and commuting,  $E_1, E_2$  are the coordinate vector fields of a local system  $(x, y)$  in which the metric is  $dx^2 + dy^2$ :  $\lambda \equiv 1$ .

*The matrix of  $S$ .* By (2.4),  $X(u) = -\langle SX - \tau JX, T \rangle$ ; since  $u \equiv 0$ ,  $\langle SX, T \rangle = \tau \langle JX, T \rangle$  for every tangent  $X$ . With  $X = E_2 = T$ :  $\langle SE_2, E_2 \rangle = \tau \langle JE_2, E_2 \rangle = 0$  (as  $JE_2 = -E_1$ ), so  $S_{22} = 0$ . With  $X = E_1$ :  $\langle SE_1, E_2 \rangle = \tau \langle JE_1, E_2 \rangle = \tau$ , so  $S_{12} = S_{21} = \tau$ . Finally  $S_{11} = 2H - S_{22} = 2H$ , and  $\det S = 0 - \tau^2 = -\tau^2$ .

$T(H) = 0$ . By (2.2),  $\nabla S$  is symmetric since  $u \equiv 0$ :  $(\nabla_X S)Y = (\nabla_Y S)X$ . With  $(X, Y) = (E_1, E_2)$ :  $(\nabla_{E_1} S)E_2 = \tau \nabla_{E_1} E_1 = 0$ , and  $(\nabla_{E_2} S)E_1 = 2(E_2 H)E_1 + 2H \nabla_{E_2} E_1 + \tau \nabla_{E_2} E_2 = 2(E_2 H)E_1$ . Equating gives  $E_2 H = T(H) = 0$ .

*The formula for  $Q_{AR}$ .* With  $\lambda = 1$ ,  $\partial_\zeta = \frac{1}{2}(E_1 - iE_2)$ :  $S\partial_\zeta = \frac{1}{2}((2H - i\tau)E_1 + \tau E_2)$ , so

$$p = \langle S\partial_\zeta, \partial_\zeta \rangle = \frac{H - i\tau}{2}, \quad A = \langle \xi, \partial_\zeta \rangle = -\frac{i}{2}.$$

Substituting into  $Q_{AR} = 2(H + i\tau)p - kA^2$ ,

$$Q_{AR} = (H + i\tau)(H - i\tau) + \frac{k}{4} = H^2 + \tau^2 + \frac{\kappa - 4\tau^2}{4} = H^2 + \frac{\kappa}{4}.$$

*Holomorphy  $\iff H$  constant.* In each of the local coordinates  $\zeta = x + iy$  above,  $Q_{AR} = H^2 + \kappa/4$  is real. If  $Q_{AR}$  is holomorphic, then on each such connected coordinate disc  $Q_{AR}$  is a real-valued holomorphic function, hence constant, since a nonconstant holomorphic function is an open map and  $\mathbb{R}$  has empty interior in  $\mathbb{C}$ . Thus  $H^2$  is locally constant, hence constant on the connected surface  $\Sigma_\gamma$ ; as  $H$  is continuous and takes values in the discrete set  $\{\pm\sqrt{H^2}\}$ ,  $H$  itself is constant. The converse is immediate from (2.10).

$2H = k_g$ . Let  $V$  be the unit tangent of  $\gamma$  and  $\tilde{N}$  the unit normal of  $\gamma$  whose horizontal lift is  $N$ , so that  $\nabla_V^M V = k_g \tilde{N}$ . The field  $E_1 = -J\xi$  is horizontal and unit, and tangent to  $\Sigma_\gamma$ ; hence its integral curves are horizontal lifts  $\tilde{\gamma}$  of  $\gamma$  parametrized by arc length, with  $d\pi(\tilde{\gamma}') = \pm V$ ; the sign plays no role below, since  $\nabla_V^M V$  is unchanged by reversing the orientation of  $\gamma$ . By the Gauss formula,  $S_{11} = \langle SE_1, E_1 \rangle = \langle \bar{\nabla}_{\tilde{\gamma}'} \tilde{\gamma}', N \rangle$ . For basic fields  $X, Y$  on  $E(\kappa, \tau)$  (horizontal lifts of vector fields on the base) one has  $\bar{\nabla}_X Y = \mathcal{H}(\bar{\nabla}_X Y) + A_X Y$ , where  $\mathcal{H}(\bar{\nabla}_X Y)$  is the horizontal lift of the covariant derivative in the base and O'Neill's tensor  $A$  is alternating on horizontal vectors [22, Ch. 7]; in particular  $A_{E_1} E_1 = 0$ , so  $\bar{\nabla}_{\tilde{\gamma}'} \tilde{\gamma}'$  is the horizontal lift of  $\nabla_V^M V$ . Therefore  $S_{11} = \langle \nabla_V^M V, d\pi(N) \rangle = \langle k_g \tilde{N}, \tilde{N} \rangle = k_g$ , and since  $S_{11} = 2H$ , we get  $2H = k_g$ .  $\square$

By Proposition 3.2,  $\det S \equiv -\tau^2 < 0$  on every Hopf tube (as  $\tau \neq 0$ ), so Proposition 3.1 applies: every closed Hopf tube is a torus. The closed-form expression  $Q_{AR} = H^2 + \kappa/4$  makes Hopf tori the natural place to look for non-CMC examples satisfying the inequality of Theorem 1.1; see Proposition 5.1.

#### 4. PROOF OF THE MAIN THEOREM

In this section  $Q_{AR}$  is not assumed holomorphic; we only use the inequality of Theorem 1.1. All norms are computed in the induced metric  $g = \lambda|d\zeta|^2$ , with  $|Q_{AR}|_g = |Q_{AR}|/\lambda$ .

For a continuous function  $f$  we say that  $f$  has a *zero of order  $m \geq 1$*  at  $\zeta_0$  if  $f(\zeta) = (\zeta - \zeta_0)^m \varphi(\zeta)$  near  $\zeta_0$ , with  $\varphi$  continuous and  $\varphi(\zeta_0) \neq 0$ . Then  $m$  equals the winding number of  $f$  around  $\zeta_0$  on small circles, so it is well defined; for a merely continuous (or  $C^1$ )  $f$  it is not to be confused with an order of vanishing of a Taylor expansion.

**Lemma 4.1** (Similarity principle). *If  $f \in C^1(U, \mathbb{C})$  satisfies  $|\partial_{\bar{\zeta}} f| \leq a|f|$  almost everywhere, with  $a \geq 0$  in  $L^p_{\text{loc}}(U)$  for some  $p > 2$  (for instance,  $a$  locally bounded), then, near each point,  $f = \Phi F$ , where  $F$  is holomorphic and  $\Phi$  is continuous and nonvanishing. Thus, on each connected component of  $U$ , a solution that is not identically zero there has isolated zeros, of positive integer order and with the same winding number as the holomorphic factor.*

This is a version of the similarity principle of Bers and Vekua [9, §10], [27]; see also [10, Lemmas 1 and 2] for the form used in [5]. We include the short proof for completeness.

*Proof.* Fix  $\zeta_0 \in U$  and a disc  $D = D(\zeta_0, r) \Subset U$ . Define

$$b(\zeta) := \begin{cases} \frac{f_{\bar{\zeta}}(\zeta)}{f(\zeta)}, & f(\zeta) \neq 0, \\ 0, & f(\zeta) = 0. \end{cases}$$

By hypothesis,  $|f_{\bar{\zeta}}(\zeta)| \leq a(\zeta)|f(\zeta)|$  for almost every  $\zeta$ , so  $|b(\zeta)| \leq a(\zeta)$  almost everywhere; at almost every point where  $f(\zeta) = 0$  the hypothesis gives  $f_{\bar{\zeta}}(\zeta) = 0 = b(\zeta)f(\zeta)$  as well. Hence  $f_{\bar{\zeta}} = bf$  almost everywhere on  $D$ , and therefore in the sense of distributions, since  $f \in C^1$ . The function  $b$  is in general only measurable (it is continuous on the open set  $\{f \neq 0\}$  and vanishes on its complement), but  $|b| \leq a$  gives  $b \in L^p(D)$  for the exponent  $p > 2$  of the hypothesis. Accordingly, all identities below involving  $b$  are understood in the weak (distributional) sense.

Since  $b \in L^p(D)$ , the inhomogeneous Cauchy–Riemann equation  $v_{\bar{\zeta}} = b$  admits, on any slightly smaller disc  $D' \Subset D$ , a solution  $v \in W^{1,p}(D')$ : the standard  $L^p$  solvability theory for  $\bar{\partial}$ . Explicitly, the Cauchy transform

$$v(\zeta) := \frac{1}{\pi} \int_D \frac{b(\zeta')}{\zeta - \zeta'} dA(\zeta')$$

satisfies  $v_{\bar{\zeta}} = b$  in the sense of distributions, and the Calderón–Zygmund  $L^p$  estimates for its derivatives give  $v \in W^{1,p}(D')$ , since  $b \in L^p(D)$ ; see [27, Ch. I] and [8, Ch. 4] for the Cauchy transform and its  $L^p$  estimates, and [14, Ch. 9] for the general  $L^p$  theory. As  $p > 2$ , the Sobolev (Morrey) embedding  $W^{1,p}(D') \hookrightarrow C^0(\overline{D'})$  (valid in the plane since  $p > 2 = \dim$ ; see [14, Ch. 7]) gives  $v$  a continuous representative on  $D'$ , used from now on.

Set  $F := e^{-v}f$  on  $D'$ . Since  $v \in W^{1,p}(D')$  with  $p > 2$ , the Sobolev chain rule gives  $e^{-v} \in W^{1,p}(D')$  with  $\partial_{\bar{\zeta}}(e^{-v}) = -e^{-v}\partial_{\bar{\zeta}}v$  almost everywhere; combined with  $f \in C^1$ , the distributional product rule for  $\bar{\partial}$  applies, and

$$F_{\bar{\zeta}} = e^{-v}f_{\bar{\zeta}} - v_{\bar{\zeta}}e^{-v}f = e^{-v}(bf) - be^{-v}f = 0$$

in the sense of distributions on  $D'$ . By Weyl’s lemma for  $\partial_{\bar{\zeta}}$ , the continuous function  $F$  is holomorphic on  $D'$ . Set  $\Phi := e^v$ : continuous and nowhere zero. Then  $f = \Phi F$  on  $D'$ , the local factorization claimed near the arbitrary point  $\zeta_0 \in U$ .

For the last statement, fix a connected component  $U_0$  of  $U$  and suppose  $f|_{U_0} \not\equiv 0$ . If  $f$  vanished on some nonempty open  $V \subset U_0$ , patching local factorizations along a chain of discs from  $V$  to any point of  $U_0$  (using the identity theorem for each local holomorphic factor  $F$ ) would force  $f \equiv 0$  on all of  $U_0$ , a contradiction. Hence  $f$  vanishes on no nonempty open subset of  $U_0$ , and on each local disc  $D'$  the holomorphic factor  $F$  is nonzero, with isolated zeros of some positive integer order  $m$ : near such a zero  $\zeta_1$ ,  $F(\zeta) =$

$(\zeta - \zeta_1)^m G(\zeta)$ ,  $G(\zeta_1) \neq 0$ . Tracking a continuous branch of  $\arg F = m \arg(\zeta - \zeta_1) + \arg G$  around a small circle about  $\zeta_1$ , the  $G$ -term contributes no net change (continuous nonvanishing  $G$  on a simply connected disc), and likewise for  $\arg \Phi$ ; so the winding number of  $f = \Phi F$  around  $\zeta_1$  equals that of  $F$  alone, namely  $m$ .  $\square$

**Corollary 4.2** (Global dichotomy). *Let  $\Sigma$  be connected and let  $Q$  be the coefficient, in local conformal parameters, of a  $C^1$  quadratic differential with  $|Q_{\bar{\zeta}}| \leq a|Q|$  almost everywhere on each chart,  $a \in L^p_{\text{loc}}$  for some  $p > 2$  (for instance a continuous). Then either  $Q \equiv 0$  on  $\Sigma$ , or the zero set of  $Q$  is discrete, with every zero of positive integer order and winding number equal to that order.*

*Proof.* By Lemma 4.1, on each chart  $U$  either  $Q \equiv 0$  or the zeros of  $Q$  in  $U$  are isolated, of positive integer order, with winding number equal to that order. Let

$$Z^\circ := \{p \in \Sigma : Q \equiv 0 \text{ on some open neighborhood of } p\}.$$

$Z^\circ$  is open by construction. It is also closed: let  $p$  be a point of the closure of  $Z^\circ$ , and fix a connected chart  $U \ni p$  meeting  $Z^\circ$ , so  $Q \equiv 0$  on a nonempty open subset of  $U$ ; applying Lemma 4.1 on  $U$ , the isolated-zeros alternative is incompatible with  $Q$  vanishing on a nonempty open set, so  $Q \equiv 0$  on all of  $U \subset Z^\circ$ , and  $p \in Z^\circ$ .

As  $\Sigma$  is connected and  $Z^\circ$  is clopen,  $Z^\circ \in \{\emptyset, \Sigma\}$ . If  $Z^\circ = \Sigma$ , then  $Q \equiv 0$  on  $\Sigma$ . If  $Z^\circ = \emptyset$ , the isolated-zeros alternative holds on every chart; the notions of isolated zero, order, and winding number are local and unaffected by the transition between overlapping charts: if  $Q = q(\zeta) d\zeta^2 = \tilde{q}(\omega) d\omega^2$  on an overlap, then  $\tilde{q}(\omega) = q(\zeta(\omega)) (d\zeta/d\omega)^2$ , and since  $d\zeta/d\omega \neq 0$  is holomorphic, this is a nonvanishing holomorphic multiple of  $q \circ \zeta$ , which changes neither the order of a zero nor its winding number. So the zero set of  $Q$  on all of  $\Sigma$  is discrete, of matching order and winding number.  $\square$

*Proof of Theorem 1.1. Step 1: a Cauchy–Riemann inequality for  $Q_{\text{AR}}$  itself.* The fundamental identity (2.10),  $(Q_{\text{AR}})_{\bar{\zeta}} = 2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\bar{\zeta}}$ , holds for every smooth immersion, with no holomorphy hypothesis on  $Q_{\text{AR}}$ . Write  $\rho := \sqrt{H^2 + \tau^2} = |H + i\tau|$ . In an isothermal parameter  $\zeta = x + iy$ , since  $H$  is real,  $|H_{\bar{\zeta}}| = |H_{\zeta}| = \frac{1}{2}\sqrt{\lambda}|dH|_g$ . By the triangle inequality,

$$|(Q_{\text{AR}})_{\bar{\zeta}}| \leq 2|p||H_{\bar{\zeta}}| + \lambda\rho|H_{\bar{\zeta}}| = (2|p| + \lambda\rho)\frac{\sqrt{\lambda}}{2}|dH|_g.$$

Substituting the hypothesis  $|dH|_g \leq h|Q_{\text{AR}}|_g = h|Q_{\text{AR}}|/\lambda$ , valid almost everywhere,

$$(4.1) \quad |(Q_{\text{AR}})_{\bar{\zeta}}| \leq \frac{h(2|p| + \lambda\rho)}{2\sqrt{\lambda}}|Q_{\text{AR}}| =: a|Q_{\text{AR}}|,$$

with  $a$  continuous on each chart when  $h$  is continuous, and  $a \in L^p_{\text{loc}}$  when  $h \in L^p_{\text{loc}}$ , since the factor  $(2|p| + \lambda\rho)/(2\sqrt{\lambda})$  is continuous. Moreover  $Q_{\text{AR}}$  is smooth, since the immersion is. Thus, on each chart,  $f = Q_{\text{AR}}$  satisfies the hypotheses of Lemma 4.1:  $f \in C^1$ , and  $a \in L^p_{\text{loc}}$  for the exponent  $p > 2$  of Theorem 1.1. When  $h$  is continuous,  $a$  is locally bounded, so  $a \in L^p_{\text{loc}}$  for every  $p > 2$  and the continuous version is the special case  $a \in L^\infty_{\text{loc}}$  of the measurable one; no further regularity of  $h$  is used.

*Step 2: the dichotomy.* By (4.1),  $Q_{\text{AR}}$  satisfies, on each chart, the hypothesis of Corollary 4.2, so either  $Q_{\text{AR}} \equiv 0$  on the connected  $\Sigma$ , or its zero set

is discrete, with every zero of positive integer order and matching winding number.

*Step 3: the second alternative is impossible on  $S^2$ .* Suppose, for contradiction,  $\mathcal{Q}_{\text{AR}} \not\equiv 0$ . On the complement of its zero set, the directions  $v$  with  $\mathcal{Q}_{\text{AR}}(v, v) = \mathcal{Q}_{\text{AR}}(d\zeta(v))^2$  real and positive define a smooth line field  $\ell$ , at angle  $\theta \equiv -\frac{1}{2} \arg \mathcal{Q}_{\text{AR}} \pmod{\pi}$  to the coordinate axis. Since  $\Sigma \cong S^2$  is compact and the zeros of  $\mathcal{Q}_{\text{AR}}$  are isolated by Step 2, there are finitely many  $q_1, \dots, q_r$  ( $r \geq 0$ ), of orders  $m_1, \dots, m_r \geq 1$ .

For a line field  $\ell$  with an isolated singularity  $q$ , we use Hopf's convention [17, Part Two, Ch. III, §1.2, p. 107]:  $\text{ind}_q(\ell) := \Delta\theta/2\pi$ , where  $\theta$  is a continuous lift of the angle between  $\ell$  and a coordinate axis along a small positively oriented circle around  $q$ ; it is an integer or a half-integer. We compute the index of  $\ell$  at each  $q_j$ . In a conformal chart  $\zeta$  centered at  $q_j = 0$ , the local factorization of Lemma 4.1 gives  $\mathcal{Q}_{\text{AR}} = \Phi_j F_j$  near  $q_j$ ,  $F_j$  holomorphic with a zero of order  $m_j$  at 0,  $\Phi_j$  continuous and nonvanishing; write  $F_j(\zeta) = \zeta^{m_j} G_j(\zeta)$ ,  $G_j(0) \neq 0$ . On a small circle, tracking a continuous branch of  $\arg \mathcal{Q}_{\text{AR}} = \arg \Phi_j + m_j \arg \zeta + \arg G_j$ , only the  $\arg \zeta$  term contributes net change,  $2\pi m_j$ ; so a continuous lift of  $\theta = -\frac{1}{2} \arg \mathcal{Q}_{\text{AR}} \pmod{\pi}$  changes by  $\Delta\theta = -\pi m_j$ , and

$$\text{ind}_{q_j}(\ell) = -\frac{m_j}{2}.$$

The Poincaré–Hopf index formula for line fields with (possibly half-integer) indices on a closed surface (see [17, Part Two, Ch. III, §2.2, p. 113]; also [16, Einleitung, Nr. 2], where indices are normalized as twice ours, so that the formula reads  $\sum j = 4(1 - g)$ ) gives

$$2 = \chi(S^2) = \chi(\Sigma) = \sum_{j=1}^r \left(-\frac{m_j}{2}\right).$$

(If  $r = 0$ ,  $\ell$  is already nonsingular on all of  $\Sigma$ , and the classical Poincaré–Hopf formula gives  $\chi(S^2) = 0$  directly—already a contradiction; the displayed identity extends this to  $r \geq 1$ .) The right side is  $\leq 0$  (empty if  $r = 0$ , else a sum of strictly negative terms), contradicting  $= 2$ . Hence  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  is impossible.

*Step 4: conclusion.* By Steps 2–3,  $\mathcal{Q}_{\text{AR}} \equiv 0$  on  $\Sigma$ , so  $|dH|_g \leq h \cdot 0 = 0$  almost everywhere; as  $dH$  is continuous,  $dH \equiv 0$ , and  $H$  is constant on connected  $\Sigma$ .  $\square$

**Remark 4.3** (The case  $\tau = 0$ ). Every step of the proof holds for any sign of  $k = \kappa - 4\tau^2$ , in particular at  $k = 0$  (the round sphere if  $\tau \neq 0$ , and  $\mathbb{R}^3$  if  $\tau = 0$ ), and no step uses  $\tau \neq 0$ ; this is why Theorem 1.1 is stated for all  $\kappa, \tau \in \mathbb{R}$ . For  $\tau = 0$  and continuous  $h$  it is the first conclusion of [5, Theorem 1.1]. The condition  $\tau \neq 0$  is used only in Proposition 5.1, through  $\det S = -\tau^2 < 0$ , and it is the setting of the companion paper [6].

**Corollary 4.4.** *Let  $\Sigma$  be a closed, connected, oriented surface of genus  $g$ , immersed in  $E(\kappa, \tau)$  and satisfying  $|dH|_g \leq h|\mathcal{Q}_{\text{AR}}|_g$  with  $h$  as in Theorem 1.1. If  $\mathcal{Q}_{\text{AR}} \not\equiv 0$ , then  $\mathcal{Q}_{\text{AR}}$  has finitely many zeros  $q_1, \dots, q_r$ , of orders  $m_1, \dots, m_r \geq 1$ , and*

$$\sum_{j=1}^r m_j = 4g - 4.$$

*In particular  $g \geq 1$ ; if  $g = 1$ , then  $\mathcal{Q}_{\text{AR}}$  has no zeros; and if  $g \geq 2$ , then  $1 \leq r \leq 4g - 4$ .*

*Proof.* Steps 1–3 of the proof of Theorem 1.1 use the topology of  $\Sigma$  only through  $\chi(\Sigma)$  and its compactness, which makes the discrete zero set of Step 2 finite; each  $m_j$  is the order of the zero  $q_j$ , so the sum counts zeros with multiplicity. For genus  $g$  they give  $2 - 2g = \chi(\Sigma) = -\frac{1}{2} \sum_j m_j$ , that is,  $\sum_j m_j = 4g - 4$ . The remaining assertions follow since each  $m_j \geq 1$ .  $\square$

**Remark 4.5.** Corollary 4.4 shows that, without any holomorphy, the zeros of  $\mathcal{Q}_{\text{AR}}$  obey the same count as those of a holomorphic quadratic differential. The count by itself does not decide whether  $H$  is constant: for  $g = 1$ , Proposition 5.1 gives zero-free non-CMC examples, while for  $g \geq 2$  zeros are forced to exist and the zero-free examples of Proposition 5.1 have no analogue. Nevertheless, the inequality does not force  $H$  to be constant in any genus  $g \geq 1$ : see Theorem 6.1 and Corollary 6.4.

## 5. SHARPNESS: NON-CMC HOPF TORI ON BERGER SPHERES

**Proposition 5.1** (Non-CMC Hopf tori satisfying the inequality). *On every Berger sphere, including the round sphere, there exist smooth, compact, oriented, non-CMC tori for which*

$$|dH|_g \leq h_0 |\mathcal{Q}_{\text{AR}}|_g$$

with a finite constant  $h_0 \geq 0$ . In particular, the conclusion of Theorem 1.1 fails for closed surfaces of genus one.

*Proof.* Every Berger sphere has base curvature  $\kappa > 0$  (round case included,  $\kappa = 4\tau^2 > 0$ ). We need a smooth, embedded, simple closed curve  $\gamma \subset S^2(\kappa)$  whose geodesic curvature  $k_g$  is non-constant. Any such curve that is not a circle will do, since the curves of constant geodesic curvature in  $S^2(\kappa)$  are circles; for definiteness we give an explicit one. Realize  $S^2(\kappa)$  as the Euclidean sphere of radius  $R = 1/\sqrt{\kappa}$  in  $\mathbb{R}^3$ . Fix  $0 < \varepsilon < \pi/4$  and set  $\beta(\varphi) := \varepsilon \cos(2\varphi)$ , so that  $|\beta(\varphi)| \leq \varepsilon < \pi/4$  for all  $\varphi$ ; define

$$\gamma(\varphi) := R(\cos \beta(\varphi) \cos \varphi, \cos \beta(\varphi) \sin \varphi, \sin \beta(\varphi)), \quad 0 \leq \varphi \leq 2\pi.$$

Since  $\cos \beta(\varphi) > 0$  throughout, the longitude  $\varphi$  is recovered uniquely from  $\gamma(\varphi)$  (as  $\arg(\gamma_1 + i\gamma_2)$ ), so  $\gamma$  is simple and closed. Let

$$e_\beta := (-\sin \beta \cos \varphi, -\sin \beta \sin \varphi, \cos \beta), \quad e_\varphi := (-\sin \varphi, \cos \varphi, 0)$$

be the unit vectors, tangent to the sphere, in the directions of increasing latitude and longitude. Then  $\gamma' = R(\beta' e_\beta + \cos \beta e_\varphi)$ , so  $|\gamma'|^2 = R^2(\beta'^2 + \cos^2 \beta) > 0$  and  $\gamma$  is a smooth, regular, embedded curve. Note that  $\langle e_\beta, \gamma' \rangle = R\beta'$ , so  $e_\beta$  is normal to  $\gamma$  only where  $\beta' = 0$ . A smooth unit normal to  $\gamma$ , tangent to the sphere and pointing north ( $\langle n, e_\beta \rangle > 0$ ), is

$$n := \frac{\cos \beta e_\beta - \beta' e_\varphi}{\sqrt{\cos^2 \beta + \beta'^2}}.$$

A standard computation in the coordinates  $ds^2 = R^2(d\beta^2 + \cos^2 \beta d\varphi^2)$  gives the geodesic curvature of  $\gamma$  with respect to  $n$ :

$$k_g = \frac{\cos \beta \beta'' + \sin \beta (\cos^2 \beta + 2\beta'^2)}{R(\cos^2 \beta + \beta'^2)^{3/2}}.$$

(For  $\beta$  constant this is  $\tan \beta/R$ , the geodesic curvature of a parallel.) At the points where  $\beta'(\varphi) = 0$  we have  $n = e_\beta$  and

$$k_g(\varphi) = \frac{\beta''(\varphi) + \sin \beta(\varphi) \cos \beta(\varphi)}{R \cos^2 \beta(\varphi)}.$$

Here  $\beta'(\varphi) = -2\varepsilon \sin(2\varphi)$  vanishes at  $\varphi = 0, \pi/2, \pi, 3\pi/2$ , with  $\beta(0) = \varepsilon$ ,  $\beta(\pi/2) = -\varepsilon$ , and  $\beta''(\varphi) = -4\varepsilon \cos(2\varphi)$ , so  $\beta''(0) = -4\varepsilon$ ,  $\beta''(\pi/2) = 4\varepsilon$ . Hence

$$k_g(0) = \frac{-4\varepsilon + \sin \varepsilon \cos \varepsilon}{R \cos^2 \varepsilon} = \frac{-4\varepsilon + \frac{1}{2} \sin(2\varepsilon)}{R \cos^2 \varepsilon},$$

$$k_g(\pi/2) = \frac{4\varepsilon - \frac{1}{2} \sin(2\varepsilon)}{R \cos^2 \varepsilon} = -k_g(0).$$

Since  $\sin(2\varepsilon) < 2\varepsilon$  for every  $\varepsilon > 0$ , the numerator  $-4\varepsilon + \frac{1}{2} \sin(2\varepsilon)$  is strictly negative, so  $k_g(0) < 0 < k_g(\pi/2) = -k_g(0)$ : in particular  $k_g(0) \neq k_g(\pi/2)$ , and  $k_g$  is non-constant on  $\gamma$ .

Let  $\Sigma_\gamma = \pi^{-1}(\gamma)$  be the Hopf tube over  $\gamma$  (§3.1), oriented by the horizontal lift  $N$  of  $n$ . Since  $\gamma$  is smooth, simple, and closed in the compact base  $S^2(\kappa)$ ,  $\pi|_{\Sigma_\gamma}$  is a smooth fiber bundle over  $\gamma \cong S^1$  with fiber  $S^1$ , so  $\Sigma_\gamma$  is a smooth, compact, connected, embedded surface without boundary. By Proposition 3.2,  $\det S \equiv -\tau^2 < 0$  on  $\Sigma_\gamma$  (as  $\tau \neq 0$ ), so Proposition 3.1 identifies  $\Sigma_\gamma$  as a torus.

Again by Proposition 3.2,  $2H = k_g$  is non-constant, so  $\Sigma_\gamma$  is not CMC, and  $\mathcal{Q}_{\text{AR}} = H^2 + \kappa/4$  in the parallel-frame coordinates where  $\lambda \equiv 1$ , so

$$|\mathcal{Q}_{\text{AR}}|_g = \frac{|\mathcal{Q}_{\text{AR}}|}{\lambda} = H^2 + \frac{\kappa}{4} \geq \frac{\kappa}{4} > 0$$

identically on  $\Sigma_\gamma$ . As  $\Sigma_\gamma$  is compact and  $H$  is smooth,  $|dH|_g$  attains a finite maximum. Set  $h_0 := \frac{4}{\kappa} \max_{\Sigma_\gamma} |dH|_g < \infty$ . Then, at every point,

$$h_0 |\mathcal{Q}_{\text{AR}}|_g \geq h_0 \cdot \frac{\kappa}{4} = \max_{\Sigma_\gamma} |dH|_g \geq |dH|_g,$$

which is the claimed inequality.

By Proposition 3.2,  $\mathcal{Q}_{\text{AR}}$  is not holomorphic on  $\Sigma_\gamma$ , since  $H$  is not constant; this does not contradict Theorem 1.1, because  $\Sigma_\gamma$  is a torus.  $\square$

Thus the genus-zero hypothesis of Theorem 1.1 cannot be dropped: the conclusion fails in genus one, even with a *constant*  $h_0$ ; genus  $g \geq 2$  is treated in Section 6.

**Remark 5.2** (Where the hypothesis has content). On any compact  $\Sigma$  on which  $\mathcal{Q}_{\text{AR}}$  has no zeros, a continuous bound  $h$  as in Theorem 1.1 exists automatically: one may take the constant

$$h := \max_{x \in \Sigma} \frac{|dH|_g(x)}{|\mathcal{Q}_{\text{AR}}|_g(x)} < \infty,$$

the maximum of a continuous function on a compact surface; this is exactly the mechanism behind Proposition 5.1. Thus the Cauchy–Riemann inequality, as a hypothesis, has geometric content only *near* the zeros of  $\mathcal{Q}_{\text{AR}}$ , which is precisely where the index computation in Step 3 of the proof of Theorem 1.1 takes place. Any extension of Theorem 1.1 to genus one must impose additional conditions that exclude the zero-free non-CMC examples of Proposition 5.1; the existence of a continuous bound  $h$  alone does not suffice, since it already holds there. For genus  $g \geq 2$ , see Section 6.

## 6. SHARPNESS IN EVERY GENUS

In genus  $g \geq 2$  zeros of  $\mathcal{Q}_{\text{AR}}$  are forced (Corollary 4.4), and one might hope that the inequality near the zeros forces  $H$  to be constant. The following perturbation argument shows that it does not.

**Theorem 6.1.** *Let  $\psi_0 : \Sigma \rightarrow E(\kappa, \tau)$ ,  $\kappa, \tau \in \mathbb{R}$ , be an immersion of a closed, connected, oriented surface of genus  $g$ , with constant mean curvature and  $\mathcal{Q}_{\text{AR}} \not\equiv 0$ . Then there are immersions  $\psi_\varepsilon$ ,  $|\varepsilon| < \varepsilon_0$ , depending smoothly on  $\varepsilon$  and coinciding with  $\psi_0$  outside a compact disc  $\overline{D} \subset \Sigma$ , such that for every  $\varepsilon \neq 0$  the mean curvature  $H_\varepsilon$  of  $\psi_\varepsilon$  is not constant and*

$$|dH_\varepsilon|_{g_\varepsilon} \leq h_\varepsilon |\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon}$$

with a constant  $h_\varepsilon$  satisfying  $0 < h_\varepsilon \leq C|\varepsilon|$ , where  $C$  depends on  $\psi_0$ ,  $\overline{D}$  and the perturbation (see the proof) but not on  $\varepsilon$ . The zeros of  $\mathcal{Q}_{\text{AR}}^\varepsilon$  are exactly those of  $\mathcal{Q}_{\text{AR}}$ , and  $\psi_\varepsilon = \psi_0$  near them. If  $\psi_0$  is an embedding, so is  $\psi_\varepsilon$  for  $\varepsilon_0$  small.

*Proof.* Since  $H_0$  is constant,  $\mathcal{Q}_{\text{AR}}$  is holomorphic by (2.10). As  $\mathcal{Q}_{\text{AR}} \not\equiv 0$  and  $\Sigma$  is connected, its zeros are isolated, hence finite. Choose a closed disc  $\overline{D} \subset \Sigma$  containing no zero of  $\mathcal{Q}_{\text{AR}}$ , so that  $|\mathcal{Q}_{\text{AR}}|_{g_0} \geq c > 0$  on  $\overline{D}$ . Fix  $\varphi \in C^\infty(\Sigma)$ ,  $\varphi \not\equiv 0$ , supported in  $D$ , and put  $\psi_\varepsilon(x) = \exp_{\psi_0(x)}(\varepsilon\varphi(x)N_0(x))$ , where  $N_0$  is the unit normal of  $\psi_0$ . For  $|\varepsilon|$  small,  $\psi_\varepsilon$  is an immersion, it coincides with  $\psi_0$  outside  $D$ , and it is an embedding if  $\psi_0$  is, since embeddings of a compact manifold form an open set in the  $C^1$  topology.

$H_\varepsilon$  is not constant. The first variation of the mean curvature under the normal variation  $\varphi N_0$  is

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} H_\varepsilon = \frac{1}{2} L\varphi, \quad L = \Delta_{g_0} + |S_0|^2 + \overline{\text{Ric}}(N_0, N_0),$$

where  $\Delta_{g_0} = \text{div } \nabla$  is the Laplacian of  $g_0$  (the trace of the Hessian, so that  $\Delta = \partial_x^2 + \partial_y^2$  for the flat metric),  $|S_0|^2 = \text{tr } S_0^2$  ( $L$  is the Jacobi operator of  $\psi_0$ ; cf. [26, p. 1129]), and the factor  $\frac{1}{2}$  comes from  $H = \frac{1}{2} \text{tr } S$  with  $S = -\nabla N$ . (For the unit sphere in  $\mathbb{R}^3$  with outward normal,  $H = -1$ , and  $\varphi \equiv 1$  gives  $\frac{1}{2} L\varphi = 1 = \frac{d}{d\varepsilon}(-1/(1+\varepsilon))|_{\varepsilon=0}$ .) Hence  $H_\varepsilon = H_0 + \frac{\varepsilon}{2} L\varphi + O(\varepsilon^2)$  uniformly on  $\Sigma$ . The function  $L\varphi$  vanishes outside  $D$ . It is not identically zero: otherwise  $\varphi$  would be a solution of the elliptic equation  $L\varphi = 0$  vanishing on the nonempty open set  $\Sigma \setminus \overline{D}$ , hence  $\varphi \equiv 0$  by unique continuation [7]. Choose  $x_1 \notin \overline{D}$  and  $x_2$  with  $L\varphi(x_2) \neq 0$ . Then  $H_\varepsilon(x_2) - H_\varepsilon(x_1) = \frac{\varepsilon}{2} L\varphi(x_2) + O(\varepsilon^2) \neq 0$  for  $0 < |\varepsilon| < \varepsilon_0$ , after shrinking  $\varepsilon_0$ .

*The inequality.* The norms  $|dH_\varepsilon|_{g_\varepsilon}$  and  $|\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon}$  are functions on  $\Sigma$  independent of the conformal parameter. The second one depends only on the metric, the second fundamental form and the tangential part of  $\xi$ , hence continuously on the 2-jet of the immersion, so  $|\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon} \geq c/2$  on  $\overline{D}$  for  $\varepsilon_0$  small. Outside  $\overline{D}$  we have  $\psi_\varepsilon = \psi_0$ , so  $H_\varepsilon = H_0$  is constant there and  $dH_\varepsilon = 0$ . Hence the inequality holds with

$$h_\varepsilon := \max_{\overline{D}} \frac{|dH_\varepsilon|_{g_\varepsilon}}{|\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon}} < \infty.$$

Finally,  $H_\varepsilon - H_0$  depends smoothly on  $\varepsilon$  in  $C^1(\Sigma)$  and vanishes at  $\varepsilon = 0$ , so  $\max_\Sigma |dH_\varepsilon|_{g_\varepsilon} \leq C'|\varepsilon|$ ; together with  $|\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon} \geq c/2$  on  $\overline{D}$  this gives  $h_\varepsilon \leq 2C'|\varepsilon|/c$ ; here  $c$  depends on  $\psi_0$  and  $\overline{D}$ , and  $C'$  on  $\psi_0$  and  $\varphi$ . Moreover  $h_\varepsilon > 0$ , since  $H_\varepsilon$  is not constant. Finally,  $|\mathcal{Q}_{\text{AR}}^\varepsilon|_{g_\varepsilon} \geq c/2$  on  $\overline{D}$ , so no zero is created there, while outside  $\overline{D}$  the immersion, and hence  $\mathcal{Q}_{\text{AR}}^\varepsilon$ , is unchanged; thus the zero set of  $\mathcal{Q}_{\text{AR}}^\varepsilon$ , with its orders, equals that of  $\mathcal{Q}_{\text{AR}}$ .

The constant  $h_\varepsilon$  is positive everywhere, in particular near the zeros. If one prefers a bound  $h$  that vanishes near the zeros, let  $\chi \in C^\infty(\Sigma, [0, 1])$  be equal to 1 on  $\overline{D}$  and to 0 on a neighborhood of the zero set of  $\mathcal{Q}_{\text{AR}}$  (which lies outside

$\overline{D})$ ; since  $dH_\varepsilon = 0$  outside  $\text{supp } \varphi \subset D$ , the inequality holds with the smooth function  $h = h_\varepsilon \chi$ , which vanishes identically near the zeros of  $Q_{\text{AR}}^\varepsilon$ .  $\square$

**Remark 6.2** (Abundance). The function  $\varphi$  in the proof is an arbitrary nonzero smooth function supported in a disc free of zeros of  $Q_{\text{AR}}$ . Hence, near every closed CMC surface with  $Q_{\text{AR}} \neq 0$ , the non-CMC immersions satisfying the inequality with a constant  $h$  form an infinite-dimensional family; they coincide with the CMC surface near the zeros of  $Q_{\text{AR}}$ . The constant  $C$  in  $h_\varepsilon \leq C|\varepsilon|$  depends on the chosen  $\varphi$ , and is not uniform over this family.

To apply Theorem 6.1 one needs closed CMC surfaces with  $Q_{\text{AR}} \neq 0$ . The next lemma shows that, in positive genus, this is automatic under a mild condition.

Lemma 6.3 below also follows from the local classification of Domínguez-Vázquez and Manzano [12, Proposition 2.4]: every CMC surface in  $E(\kappa, \tau)$  with vanishing differential is congruent to an open subset of one of four explicit complete surfaces, and when  $4H^2 + \kappa > 0$  only the rotational surface  $S_{H, \kappa, \tau}$ , which is then an  $H$ -sphere, occurs (the screw-motion and parabolic examples require  $4H^2 + \kappa < 0$ , and the vertical cylinders  $4H^2 + \kappa = 0$ ). Their proof uses, for  $\kappa \neq 4\tau^2$ , the same strategy as ours, namely Daniel's sister correspondence and the Abresch–Rosenberg classification in the product spaces. We include our short proof, which passes from the local classification to the compact surface by an explicit analytic continuation and covering argument, for completeness.

**Lemma 6.3** (Nonvanishing in positive genus). *Let  $\psi : \Sigma \rightarrow E(\kappa, \tau)$ ,  $\kappa, \tau \in \mathbb{R}$ , be an immersion of a closed, connected, oriented surface of genus  $g \geq 1$  with constant mean curvature  $H$ , and suppose that  $\kappa + 4H^2 > 0$ . Then  $Q_{\text{AR}} \neq 0$ .*

*Proof.* Suppose  $Q_{\text{AR}} \equiv 0$ . Let  $\tilde{\Sigma}$  be the universal cover of  $\Sigma$ , with the lifted immersion  $\tilde{\psi}$  and the lifted metric. Since  $g \geq 1$ ,  $\tilde{\Sigma}$  is noncompact, and it is complete, being the universal cover of a compact Riemannian surface.

*The case  $k = \kappa - 4\tau^2 = 0$ .* Then  $Q_{\text{AR}} = 2(H + i\tau)p$ . If  $\tau \neq 0$ , or if  $\tau = 0$  and  $H \neq 0$ , then  $H + i\tau \neq 0$ , so  $p \equiv 0$  and  $\Sigma$  is a closed totally umbilic surface of constant curvature in  $S^3(\tau^2)$  or, when  $\tau = \kappa = 0$ , in  $\mathbb{R}^3$ ; such a surface is a round sphere, a contradiction. The remaining case  $\tau = \kappa = H = 0$  is excluded by  $\kappa + 4H^2 > 0$ .

*The case  $k \neq 0$ .* Then  $E(\kappa, \tau)$  and  $M^2(k) \times \mathbb{R}$  both have 4-dimensional isometry groups. Put  $\kappa' = k$ ,  $\tau' = 0$  and  $H' = \sqrt{H^2 + \tau^2}$ , so that  $\kappa' - 4\tau'^2 = \kappa - 4\tau^2$  and  $\tau'^2 + H'^2 = \tau^2 + H^2$ ; the condition  $H^2 \geq \tau'^2 - \tau^2$  of [11, Theorem 5.2] holds trivially. Let  $\theta \in \mathbb{R}$  satisfy  $\tau' + iH' = e^{i\theta}(\tau + iH)$ . By Daniel's sister correspondence [11, Proposition 5.1 and Theorem 5.2], applied to the simply connected surface  $\tilde{\Sigma}$ , there is an isometric immersion  $\tilde{\psi}' : \tilde{\Sigma} \rightarrow M^2(k) \times \mathbb{R}$  with constant mean curvature  $H'$ , whose shape operator and tangential vertical field are  $S' = e^{\theta J}(S - HI) + H'I$  and  $T' = e^{\theta J}T$ . Since  $J\partial_\zeta = i\partial_\zeta$ , this gives  $p' = e^{-i\theta}p$  and  $A' = e^{-i\theta}A$ , while  $\tau' + iH' = e^{i\theta}(\tau + iH)$  is equivalent to  $H' + i\tau' = e^{-i\theta}(H + i\tau)$ . As  $k' = \kappa' - 4\tau'^2 = k$ ,

$$Q'_{\text{AR}} = 2(H' + i\tau')p' - kA'^2 = e^{-2i\theta}(2(H + i\tau)p - kA^2) = e^{-2i\theta}Q_{\text{AR}} \equiv 0.$$

The metric induced by  $\tilde{\psi}'$  is the complete metric of  $\tilde{\Sigma}$ , and

$$\kappa' + 4H'^2 = \kappa - 4\tau^2 + 4H^2 + 4\tau^2 = \kappa + 4H^2 > 0.$$

If  $H' = 0$ , then  $H = \tau = 0$ , so  $k = \kappa > 0$  and  $Q_{\text{AR}} = -\kappa A^2$ ; thus  $A \equiv 0$ , so  $T \equiv 0$  and the surface is everywhere orthogonal to  $\xi = \partial_t$ . Hence  $t$  is locally constant on it, and the immersion of the closed connected surface  $\Sigma$  is a covering of a slice  $S^2(\kappa) \times \{t\}$ ; as  $S^2$  is simply connected,  $\Sigma$  is a sphere, a contradiction. If  $H' \neq 0$ , the Abresch–Rosenberg classification of CMC surfaces with vanishing differential in the product spaces [1, Propositions 4.1 and 4.3] (the local part of [1, Theorem 3]) applies: since  $4H'^2 + \kappa' > 0$  and  $H' \neq 0$ , every point of  $\tilde{\Sigma}$  has a neighborhood on which  $\tilde{\psi}'$  is congruent to an open piece of one of the embedded rotationally invariant spheres  $S_{H'}^2$ . We claim that a single copy works for all of  $\tilde{\Sigma}$ . Fix  $x_0 \in \tilde{\Sigma}$  and a copy  $S_0$  of  $S_{H'}^2$  containing the image of a neighborhood of  $x_0$ , and let  $W$  be the set of points of  $\tilde{\Sigma}$  having a neighborhood whose image lies in  $S_0$ . Then  $W$  is open and nonempty. It is also closed: if  $x$  is in the closure of  $W$ , a connected neighborhood  $V$  of  $x$  is mapped into some copy  $S_1$ , and  $V$  meets  $W$ , so  $S_0 \cap S_1$  contains the image of a nonempty open subset of  $V$ , which is an open subset of both spheres ( $\tilde{\psi}'$  being an immersion). Two congruent copies of  $S_{H'}^2$  are connected real analytic surfaces (CMC surfaces in the real analytic manifold  $M^2(k) \times \mathbb{R}$  are real analytic), so if they share a nonempty open subset they coincide; thus  $S_1 = S_0$  and  $x \in W$ . As  $\tilde{\Sigma}$  is connected,  $W = \tilde{\Sigma}$ , and  $\tilde{\psi}'(\tilde{\Sigma}) \subset S_0$ . Then  $\tilde{\psi}'$ , viewed as a map into  $S_0$  with its induced metric, is a local isometry between surfaces; since  $\tilde{\Sigma}$  is complete and  $S_0$  is connected, it is a covering map [20, Ch. IV, §4]. As  $S_0$  is simply connected, it is a diffeomorphism, so  $\tilde{\Sigma}$  is compact, a contradiction.  $\square$

**Corollary 6.4.** *In the round sphere  $S^3(\tau^2)$  and in every Berger sphere, for every  $g \geq 1$  there are compact embedded non-CMC surfaces of genus  $g$  satisfying  $|dH|_g \leq h_0|Q_{\text{AR}}|_g$  with a constant  $h_0$ , which can be taken arbitrarily small. In particular, the genus-zero hypothesis of Theorem 1.1 cannot be replaced by any other genus.*

*Proof.* Torralbo [25, Theorem 1] constructed, in every Berger sphere, compact embedded minimal surfaces of every genus  $g \geq 0$ ; in the round sphere one may take the Clifford torus and Lawson’s surfaces  $\xi_{g,1}$  [21]. In all these spaces  $\kappa > 0$ , so  $\kappa + 4H^2 > 0$  for minimal surfaces, and Lemma 6.3 gives  $Q_{\text{AR}} \neq 0$  for  $g \geq 1$ . Theorem 6.1 now gives the embedded non-CMC surfaces, with  $h_\varepsilon \leq C|\varepsilon|$ .  $\square$

**Remark 6.5.** For  $g = 1$  explicit examples are available without Lemma 6.3: the Hopf torus over a circle of  $S^2(\kappa)$  is CMC with  $Q_{\text{AR}} = H^2 + \kappa/4 > 0$  (Proposition 3.2; these are the tori of [24, Theorem 1(ii)]). Among them is the Clifford torus  $\{|z|^2 = |w|^2 = \frac{1}{2}\}$ , the Hopf torus over a great circle, which is minimal in every Berger sphere [25, §3] and has  $Q_{\text{AR}} = \kappa/4 > 0$ ; and Proposition 5.1 gives non-CMC Hopf tori directly. In the round sphere,  $Q_{\text{AR}} \equiv 0$  means  $p \equiv 0$ , so Lemma 6.3 is elementary there. For  $\kappa \neq 4\tau^2$  the proof above reduces, through the Daniel correspondence, to the local classification in the product spaces [1]. A classification of complete CMC surfaces in  $E(\kappa, \tau)$  with  $Q_{\text{AR}} \equiv 0$  is announced in [2, p. 18], where it is stated, with the proof deferred, only for spheres [2, Theorem 6]; it is carried out in [13, Lemma 6.1 and Theorem 2.3] and in [12, Proposition 2.4]. We do not rely on [13, Lemma 6.1], since the assertion there that such surfaces with  $4H^2 + \kappa > 0$  have positive Gauss curvature does not hold in part of this range. Indeed, at a point where  $u^2 = 1$  one has  $A = 0$ , hence  $p = 0$  when  $H + i\tau \neq 0$ , and (2.1) gives

$K = H^2 + \kappa - 3\tau^2$ ; and at a point where  $u = 0$  one has  $|p| = |k|\lambda/(8\sqrt{H^2 + \tau^2})$ , so that  $K = H^2 + \tau^2 - k^2/(16(H^2 + \tau^2))$ . Thus the rotational CMC spheres of  $\mathbb{H}^2(-1) \times \mathbb{R}$  with  $1/2 < H < 1$  have  $K < 0$  at their poles, and those of  $S^2(1) \times \mathbb{R}$  with  $0 < H < 1/2$  have  $K < 0$  along their equator. This concerns only the sign of  $K$ ; the classification itself is not affected.

**Remark 6.6** (Euclidean space). Theorem 6.1 and Lemma 6.3 also apply in  $E(0,0) = \mathbb{R}^3$ , where  $\mathcal{Q}_{\text{AR}} = 2Hpd\zeta^2$  and the condition  $\kappa + 4H^2 > 0$  reads  $H \neq 0$ . Wente's tori [28] and Kapouleas' closed CMC surfaces of every genus  $g \geq 3$  [18, Theorem 2.7 and Examples 2.8] and, by fusing Wente tori, of every genus  $g \geq 2$  [19, Main Theorem 7.7] therefore yield, by Theorem 6.1, immersed non-CMC surfaces of every genus  $g \geq 1$  in  $\mathbb{R}^3$  satisfying the inequality with  $h$  constant and arbitrarily small. They cannot be embedded perturbations of embedded CMC surfaces, since by Alexandrov's theorem the only closed embedded CMC surfaces in  $\mathbb{R}^3$  are round spheres.

**Remark 6.7** (What a rigidity theorem in positive genus would require). The examples of Theorem 6.1 coincide with a CMC surface near the zeros of  $\mathcal{Q}_{\text{AR}}$  and satisfy the inequality with  $h$  constant and arbitrarily small, or with a smooth  $h$  vanishing identically near those zeros (see the end of the proof of Theorem 6.1). Consequently, no condition that is satisfied by a closed CMC surface with  $\mathcal{Q}_{\text{AR}} \neq 0$  and depends only on the immersion near the zeros of  $\mathcal{Q}_{\text{AR}}$  (such as a prescribed order of vanishing of  $h$  there), and no positive smallness threshold on  $h$ , can force  $H$  to be constant on a closed surface of positive genus in the round sphere or in a Berger sphere. A rigidity statement in positive genus must therefore involve a global condition of a different nature. Two directions for further investigation:

- (a) In  $\text{Nil}_3$  and  $\widetilde{PSL}_2(\mathbb{R})$ , where  $\kappa \leq 0$ , Theorem 6.1 requires a closed CMC surface of positive genus with  $\mathcal{Q}_{\text{AR}} \neq 0$ ; by Lemma 6.3 it suffices that  $4H^2 > -\kappa$ . For which genera do such surfaces exist?
- (b) Is there a natural global condition under which the inequality forces  $H$  to be constant in genus  $g \geq 1$ ? Such a condition cannot be a bound on the size of the quotient  $R := |dH|_g/|\mathcal{Q}_{\text{AR}}|_g$  alone (set  $R = 0$  where  $dH = 0$ ): for the surfaces of Theorem 6.1,  $\sup R_\varepsilon \leq C|\varepsilon|$ , so  $\int_\Sigma R_\varepsilon^q dA_\varepsilon \rightarrow 0$  for every  $q \geq 1$ , whereas  $\int_\Sigma K_\varepsilon dA_\varepsilon = 4\pi(1 - g)$  is fixed.

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The authors declare that they have no conflict of interest.

## AI DISCLOSURE

A generative AI tool was used to assist with language editing, organization of the exposition, and discussion and checking of selected calculations. The authors take full responsibility for the mathematical statements, proofs, and references.

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