

ONLY CONSTANT MEAN CURVATURE SURFACES IN $E(\kappa, \tau)$, $\tau \neq 0$, HAVE A HOLOMORPHIC ORIGINAL ABRESCH–ROSENBERG DIFFERENTIAL

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ABSTRACT. For immersed surfaces in the homogeneous spaces $E(\kappa, \tau)$ with $\tau \neq 0$, we prove that the original Abresch–Rosenberg differential $[2(H + i\tau)p - (\kappa - 4\tau^2)A^2]d\zeta^2$ is holomorphic if and only if the mean curvature H is constant; here $p d\zeta^2$ is the Hopf differential and $A = \langle \xi, \partial_\zeta \rangle$. The result is local: no completeness, compactness or analyticity is assumed, and there are no exceptional surfaces. Where $dH \neq 0$, holomorphy forces the shape operator into the normal form $\begin{pmatrix} 0 & \tau \\ \tau & 2H \end{pmatrix}$ in the frame $(\nabla H/|\nabla H|, J\nabla H/|\nabla H|)$. The round case $\kappa = 4\tau^2$ is settled by an elementary argument; for $\kappa \neq 4\tau^2$ the structure equations become a rational system whose solutions are algebraic, and an embedding into a field of Puiseux series excludes every branch. In $M^2(\kappa) \times \mathbb{R}$, $\kappa \neq 0$, where the statement fails, we show that a non-CMC surface with holomorphic original differential has nowhere vanishing dH and constant angle function, and belongs to an explicit family; in particular, closed surfaces with holomorphic original differential are CMC there. We also show that the Hopf-cylinder alternative in a theorem of Dorfmeister, Inoguchi and Kobayashi contains no non-CMC surface.

1. INTRODUCTION AND MAIN RESULTS

Hopf’s differential is holomorphic precisely on constant mean curvature (CMC) surfaces in three-dimensional space forms; its vanishing on a topological sphere gives Hopf’s sphere theorem [22, Part Two, Ch. VI, §1.4, Lemma 2.2 and Theorems 2.1 and 2.6, pp. 137–140]. Abresch and Rosenberg

2020 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42, 53C30, 53C05, 12H05, 13P15.

Key words and phrases. Abresch–Rosenberg differential; Hopf differential; constant mean curvature; homogeneous 3-manifolds; algebraic functions; Puiseux series; resultants; differential fields; connections with torsion.

Use of generative AI. Claude (Anthropic) was used for English-language editing, for the typesetting of the tables, whose content was produced by the authors, and for adapting the L^AT_EX source to the journal’s template. It was also used to discuss and check selected calculations. All mathematical content, statements, proofs, and references are the authors’ own, and the authors take full responsibility for them.

H. Alencar was partially supported by the Brazilian National Council for Scientific and Technological Development – CNPq, Brazil (grant 303118/2022-9), and by the Coordination for the Improvement of Higher Education Personnel – CAPES, Brazil (Finance Code 001).

extended this construction to CMC surfaces in $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ [1, (1)–(2) and Theorem 1, p. 142] and, more generally, in $E(\kappa, \tau)$ [2, Theorems 4 and 5, pp. 16–17].

For $\tau \neq 0$, $E(\kappa, \tau)$ is the simply connected homogeneous Riemannian 3-manifold fibering over $M^2(\kappa)$ by a Riemannian submersion with geodesic fibers, whose unit vertical Killing field ξ satisfies $\bar{\nabla}_X \xi = \tau X \times \xi$. The signs $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$ give $\widetilde{PSL}_2(\mathbb{R})$, Nil_3 , and Berger spheres, respectively [11, §2, p. 91]; see also [26]. For $\tau = 0$, $E(\kappa, 0)$ is the product $M^2(\kappa) \times \mathbb{R}$ and $\xi = \partial_t$ is parallel. Write $k = \kappa - 4\tau^2$; for $\tau \neq 0$ the case $k = 0$ is the round sphere $S^3(\tau^2)$.

Throughout, a *surface* is a smooth, connected, oriented immersion of a 2-manifold Σ into $E(\kappa, \tau)$, and Σ is *closed* if it is compact without boundary. Let N be the unit normal, $S = -\bar{\nabla}N$ the shape operator, $H = \frac{1}{2} \text{tr } S$ the mean curvature, $u = \langle N, \xi \rangle$ the *angle function*, and $T = \xi - uN$ the tangent part of ξ . In a conformal parameter ζ , put $g = \lambda |d\zeta|^2$, $p = \langle S\partial_\zeta, \partial_\zeta \rangle$, and $A = \langle \xi, \partial_\zeta \rangle$. The *original* Abresch–Rosenberg differential is

$$(1.1) \quad \mathcal{Q}_{\text{AR}} = Q_{\text{AR}} d\zeta^2, \quad Q_{\text{AR}} = 2(H + i\tau)p - kA^2;$$

in the product spaces $Q_{\text{AR}} = 2Hp - \kappa A^2$. Abresch and Rosenberg proved that $\mathcal{Q}_{\text{AR}}$ is holomorphic on every CMC surface [1, Theorem 1, p. 142], [2, Theorems 4 and 5, pp. 16–17]. Our main result is the converse, without any global hypothesis.

Theorem 1.1. *Let $\tau \neq 0$ and $\kappa \in \mathbb{R}$. A surface in $E(\kappa, \tau)$ has holomorphic $\mathcal{Q}_{\text{AR}}$ if and only if its mean curvature is constant.*

No completeness, compactness, or analyticity is assumed, and the round case $\kappa = 4\tau^2$ is included; in particular, for $\tau \neq 0$ there are no local non-CMC examples. The hypothesis $\tau \neq 0$ is essential: in every product space $M^2(\kappa) \times \mathbb{R}$ there are non-CMC surfaces with holomorphic $\mathcal{Q}_{\text{AR}}$ (Proposition 6.2). For $\kappa \neq 0$ we determine all of them.

Theorem 1.2. *Let $\kappa \neq 0$, and let Σ be a surface in $M^2(\kappa) \times \mathbb{R}$ with holomorphic $\mathcal{Q}_{\text{AR}}$ and nonconstant mean curvature. Then dH vanishes nowhere on Σ , and:*

- (a) *the angle function is a constant $u_0 \notin \{0, \pm 1\}$;*
- (b) *∇H is a principal direction with principal curvature 0, and T is parallel to ∇H ;*
- (c) $|\nabla H| = \frac{|u_0| |4H^2 + \kappa(1 - u_0^2)|}{2\sqrt{1 - u_0^2}};$
- (d) *Σ is locally congruent to one of the explicit constant angle surfaces of Proposition 6.1, with the same u_0 ; up to local congruence it is determined by κ , u_0 and the sign of $4H^2 + \kappa(1 - u_0^2)$.*

Items (a)–(c) and the nonvanishing of dH hold on all of Σ ; item (d) is a local statement.

The surfaces in (d) are written down explicitly in Proposition 6.1, and Theorem 6.3 identifies Σ with one of them. For $\kappa = \pm 1$, those for which $4H^2 + \kappa(1 - u_0^2) > 0$ are, after a change of conformal parameter, the surfaces of Proposition 6.2; those with $r > 0$, hence $H < 0$, are the examples of [20, Example 3.1], and for $\kappa > 0$ the family continues smoothly across the curve $H = 0$; the new content is that constant angle is necessary and that this family exhausts all possibilities, with no completeness assumption. Since the mean curvature of a closed surface has a critical point, Theorem 1.2 gives:

Corollary 1.3. *Let $\kappa \neq 0$. Every closed surface in $M^2(\kappa) \times \mathbb{R}$ with holomorphic Q_{AR} has constant mean curvature.*

The implication $CMC \Rightarrow$ holomorphic is due to Abresch and Rosenberg [1, Theorem 1, p. 142], [2, Theorems 4 and 5, pp. 16–17]. For the converse, in Nil₃ Berdinsky and Taimanov [7, Proposition 3, p. 1014] proved that the normalized differential $\tilde{p} = Q_{AR}/(2H + i)$ is holomorphic only on CMC surfaces, and Dorfmeister, Inoguchi and Kobayashi [15, Theorem A.1, p. 441] showed that a surface with holomorphic original differential has constant mean curvature or is a Hopf cylinder; we show in Section 2 that the second alternative reduces to the first. For the normalized differential $Q_{FM} = Q_{AR}/(H + i\tau)$, Fernández and Mira [20, Theorem 1.1, p. 282] obtained a characterization in $\widetilde{PSL}_2(\mathbb{R})$, $H^2 \times \mathbb{R}$ and the Berger spheres with $0 < \kappa \leq 8\tau^2$, with exceptional non-CMC surfaces when the base curvature is negative; the cases $S^2 \times \mathbb{R}$ and $\kappa > 8\tau^2$ are left open in [20, §1, p. 282]. To our knowledge, Theorem 1.1 is the first characterization of CMC surfaces by the original differential valid for every κ and every $\tau \neq 0$, and it admits no exceptional surfaces. Under compactness, Fernández and Mira proved that a compact surface in $E(\kappa, \tau)$ with holomorphic Q_{FM} , and with nonzero Euler characteristic if $\tau \neq 0$, has constant mean curvature [20, Remark (3), p. 282, and §4, p. 289], and they note, referring to [8], that for $\tau = 0$ the same holds for the original differential [20, p. 289]; in the product spaces, Cavalcante and Lira [8, pp. 172–173] showed that on the set where $dH \neq 0$ a surface with holomorphic original differential has zero extrinsic curvature, with ∇H a principal direction of principal curvature 0, and observed that non-CMC examples must be noncompact. These arguments use the topology of the surface, whereas Theorem 1.1 is local; by Proposition 6.2, no local statement of this kind can hold when $\tau = 0$. Table 2 in Section 2 compares these results.

For closed surfaces, Theorem 1.1 and the degree of the bundle of quadratic differentials give:

Corollary 1.4. *Let $\tau \neq 0$, and let Σ be a closed surface of genus g in $E(\kappa, \tau)$ with holomorphic Q_{AR} . Then H is constant, and:*

- (i) *if $g = 0$, then $Q_{AR} \equiv 0$ and Σ is a rotationally invariant CMC sphere;*

- (ii) if $g = 1$, then Q_{AR} either vanishes identically or has no zeros;
- (iii) if $g \geq 2$ and $Q_{AR} \not\equiv 0$, then Q_{AR} has exactly $4g - 4$ zeros, counted with multiplicity.

The rotational invariance in (i) follows from [2, Theorem 6, p. 18] (proof deferred there), [17, Lemma 6.1 and Theorem 2.3] and [16, Proposition 2.4]. We do not claim that these spheres are embedded: in some Berger spheres they are only immersed [27, p. 594, and Corollary 1, p. 601].

1.1. Strategy and organization. Section 2 compares the normalizations of the Abresch–Rosenberg differential used in the literature and the results closest to ours. Section 3 fixes the conventions, compares the holomorphy conditions of Q_{AR} and Q_{FM} (Proposition 3.1), and performs the local reduction: if Q_{AR} is holomorphic and $\Omega = \{dH \neq 0\} \neq \emptyset$, then in the frame $e_1 = \nabla H/|\nabla H|$, $e_2 = Je_1$,

$$[S]_{(e_1, e_2)} = \begin{pmatrix} 0 & \tau \\ \tau & 2H \end{pmatrix}, \quad \det S = -\tau^2, \quad K = ku^2,$$

where K denotes the intrinsic curvature (Proposition 3.3). Geometrically, N is parallel along ∇H for a metric connection D which preserves ξ and whose curvature is k times a rotation about ξ (§3.1); D is flat exactly in the round case $k = 0$. Section 4 settles the round case by an elementary argument. Section 5 treats $k \neq 0$: the structure equations become a rational system whose solutions are algebraic, and an embedding into a field of Puiseux series excludes every branch. Section 6 treats the product spaces, Section 7 proves Corollary 1.4, and the Appendix contains the computation of the coefficients of the compatibility identity and facts on derivations.

2. NORMALIZATIONS AND RELATED RESULTS

The Fernández–Mira differential [20, §2, pp. 283–285, and Remark 2.6, p. 285] is

$$(2.1) \quad Q_{FM} = 2p - \frac{k}{H + i\tau} A^2, \quad Q_{AR} = (H + i\tau)Q_{FM}.$$

Here Q_{FM} is defined where $H + i\tau \neq 0$, that is, everywhere if $\tau \neq 0$ and on $\{H \neq 0\}$ if $\tau = 0$. The factor $H + i\tau$ is constant on the CMC class, and nonzero there unless $\tau = H = 0$; but off the CMC class the two holomorphy conditions differ (Proposition 3.1). In the round case $Q_{AR} = 2(H + i\tau)p$, not merely the Hopf coefficient p . Table 1 relates the coefficients used in the literature.

For Nil₃, Dorfmeister, Inoguchi and Kobayashi [15, Theorem A.1, p. 441] prove that if $B = Q_{AR}/4$ is holomorphic, the surface is CMC or a Hopf cylinder, and [15, p. 411] presents the Hopf cylinder as a non-CMC example. For the original differential the second alternative contains no non-CMC surface: on a Hopf cylinder $Q_{AR} = H^2 + \kappa/4$ depends only on the arclength of the base curve (Remark 3.2), so holomorphy forces H constant. Table 2 summarizes

Reference	Coefficient used	Relation to Q_{AR}
Abresch–Rosenberg [1, p. 142], [2, pp. 16–17]	$Q_{\text{AR}} = 2(H+i\tau)p - kA^2$	Q_{AR} itself
Fernández–Mira [20, pp. 284–285]	$Q_{\text{FM}} = 2p - \frac{k}{H+i\tau}A^2$	$Q_{\text{AR}} = (H+i\tau)Q_{\text{FM}}$
Torralbo [27, p. 594] ($\kappa \neq 4\tau^2$)	$\Theta = p - \frac{k}{2(H+i\tau)}A^2 = \frac{1}{2}Q_{\text{FM}}$	$Q_{\text{AR}} = 2(H+i\tau)\Theta$
Berdinsky– Taimanov [7, p. 1014] (Nil_3)	$\tilde{p} = p + \frac{A^2}{2H+i}$	$Q_{\text{AR}} = (2H+i)\tilde{p}$
Dorfmeister– Inoguchi–Kobayashi [15, eq. (3.9), p. 420] (Nil_3)	$B = \frac{1}{4}(2H+i)\tilde{A}$	$Q_{\text{AR}} = 4B$

TABLE 1. Coefficients used for the Abresch–Rosenberg differential in the literature, translated into the notation and sign conventions of the present paper, and their relation to the original differential $Q_{\text{AR}} = 2(H+i\tau)p - kA^2$ of (1.1). For Nil_3 the normalization is $\kappa = 0$, $\tau = \frac{1}{2}$, so $k = -1$; $\tilde{A}$ is the notation of [15].

the results closest to ours; by Proposition 3.1, the holomorphy conditions of Q_{AR} and Q_{FM} differ off the CMC class, and so do their exceptional surfaces (Section 6).

Daniel’s sister correspondence [11, Theorem 5.2, p. 114] multiplies Q_{AR} by a unimodular constant but is defined only for CMC surfaces, so it gives no information on the converse. For a survey see [21]; for a Codazzi pair whose $(2, 0)$ -part is Q_{AR} , see [18].

The companion paper [6, Theorem 1.1, p. 2] shows that the inequality $|dH|_g \leq h|Q_{\text{AR}}|_g$, with $h \geq 0$ continuous (or in L^p_{loc} for some $p > 2$), forces $Q_{\text{AR}} \equiv 0$ and H constant on topological spheres in every $E(\kappa, \tau)$, the round case $\kappa = 4\tau^2$ included; for $\tau = 0$ and for $\tau \neq 0$, $\kappa \neq 4\tau^2$, this conclusion is due to [5, Theorem 1.1, p. 284] and [4, Theorem 1, p. 526]. That sphere theorem uses the topology of the sphere and does not yield the local converse proved here. Moreover, the genus-zero hypothesis cannot be replaced by any other genus: in the round sphere and in every Berger sphere there are compact embedded non-CMC surfaces of every genus $g \geq 1$ satisfying the inequality with a constant h [6, Corollary 6.4, p. 15].

3. GEOMETRIC FORMALISM AND LOCAL REDUCTION

Conventions. We retain the notation of the introduction, including $k = \kappa - 4\tau^2$ and the round interpretation when $k = 0$. The orientation of $E(\kappa, \tau)$

Result	Space	Differential	Setting	Conclusion
[1], [2]	$E(\kappa, \tau)$	Q_{AR}	local	H constant $\Rightarrow$ holomorphic
[7, Prop. 3]	Nil_3	$Q_{AR}/(2H+i)$	local	holomorphic $\Rightarrow$ CMC
[15, Thm. A.1]	Nil_3	Q_{AR}	local	holomorphic $\Rightarrow$ CMC or Hopf cylinder
[20, Thm. 1.1]	$\widetilde{PSL}_2(\mathbb{R})$, $H^2 \times \mathbb{R}$, Berger spheres with $0 < \kappa \leq 8\tau^2$	Q_{FM}	local	holomorphic $\Rightarrow$ CMC, or exceptional non-CMC surfaces if $\kappa < 0$
[20, Rem. (3)]	$E(\kappa, \tau)$ (with $\chi(\Sigma) \neq 0$ if $\tau \neq 0$)	Q_{FM}	compact	holomorphic $\Rightarrow$ CMC
[20, p. 289], [8]	$M^2(\kappa) \times \mathbb{R}$	Q_{AR}	compact	holomorphic $\Rightarrow$ CMC
Theorem 1.1	$E(\kappa, \tau)$, $\tau \neq 0$, all κ	Q_{AR}	local	holomorphic $\Leftrightarrow$ CMC
Theorem 1.2	$M^2(\kappa) \times \mathbb{R}$, $\kappa \neq 0$	Q_{AR}	local	non-CMC $\Rightarrow$ constant angle, explicit family

TABLE 2. Results closest to the present paper. “Local” means that no completeness or compactness is assumed.

is the one for which $\overline{\nabla}_X \xi = \tau X \times \xi$; the rotation of the tangent plane is $JX = N \times X$; conformal parameters $\zeta = x + iy$ are positively oriented, with $\partial_\zeta = \frac{1}{2}(\partial_x - i\partial_y)$; for any connection the curvature is $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$; and K is the intrinsic (Gaussian) curvature of the induced metric. Two surfaces are *locally congruent* if every point of the first has a neighborhood that an isometry of the ambient space carries onto an open subset of the second. The behavior of these conventions under homotheties and under the change of orientation $\tau \mapsto -\tau$ is recorded before (3.16).

Daniel’s structure equations [11, Corollary 3.2 and Proposition 3.3, pp. 97–98; Theorem 4.3, p. 101] are

$$(3.1) \quad K = \det S + \tau^2 + ku^2,$$

$$(3.2) \quad (\nabla_X S)Y - (\nabla_Y S)X = ku(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

$$(3.3) \quad \nabla_X T = u(SX - \tau JX),$$

$$(3.4) \quad X(u) = -\langle SX - \tau JX, T \rangle, \quad |T|^2 + u^2 = 1.$$

Here K is the intrinsic curvature. In conformal coordinates,

$$(3.5) \quad K = -\frac{\Delta_0 \log \lambda}{2\lambda}, \quad \det S = H^2 - \frac{4|p|^2}{\lambda^2}, \quad \Delta_0 = 4\partial_\zeta \partial_{\bar{\zeta}}.$$

The first identity is Liouville's curvature formula for isothermal parameters, see [14, §4-3, Exercise 1]. For the second, with $e_1 = \partial_x/\sqrt{\lambda}$, $e_2 = \partial_y/\sqrt{\lambda}$ and $S_{ij} = \langle Se_i, e_j \rangle$, bilinearity and $\partial_\zeta = \frac{\sqrt{\lambda}}{2}(e_1 - ie_2)$ give $p = \frac{\lambda}{4}(S_{11} - S_{22} - 2iS_{12})$, hence $4|p|^2/\lambda^2 = \frac{1}{4}(S_{11} - S_{22})^2 + S_{12}^2 = H^2 - \det S$. In particular, $p = 0$ precisely at umbilic points.

The fundamental equations [20, Theorem 2.3, p. 283] are

$$(3.6) \quad A_{\bar{\zeta}} = \frac{u\lambda}{2}(H + i\tau), \quad p_{\bar{\zeta}} = \frac{\lambda}{2}(H_{\zeta} + kuA),$$

$$(3.7) \quad u_{\zeta} = -(H - i\tau)A - \frac{2p}{\lambda}\bar{A}, \quad 4|A|^2 = \lambda(1 - u^2).$$

They follow from (3.2)–(3.4) by complex bilinear extension, using $J\partial_\zeta = i\partial_{\bar{\zeta}}$, $\langle S\partial_\zeta, \partial_{\bar{\zeta}} \rangle = \lambda H/2$, $T = \frac{2\bar{A}}{\lambda}\partial_\zeta + \frac{2A}{\lambda}\partial_{\bar{\zeta}}$ and the Christoffel symbols of the conformal metric,

$$(3.8) \quad \nabla_{\partial_\zeta} \partial_\zeta = (\log \lambda)_\zeta \partial_\zeta, \quad \nabla_{\partial_\zeta} \partial_{\bar{\zeta}} = \nabla_{\partial_{\bar{\zeta}}} \partial_\zeta = 0.$$

For instance, $A_{\bar{\zeta}} = \langle \nabla_{\partial_{\bar{\zeta}}} T, \partial_\zeta \rangle$ with $\nabla_{\partial_{\bar{\zeta}}} T = u(S\partial_{\bar{\zeta}} + i\tau\partial_{\bar{\zeta}})$ by (3.3), and $p_{\bar{\zeta}}$ is obtained by pairing the complexified Codazzi equation $(\nabla_{\zeta} S)\partial_{\bar{\zeta}} - (\nabla_{\bar{\zeta}} S)\partial_\zeta = ku(\bar{A}\partial_\zeta - A\partial_{\bar{\zeta}})$ with $\partial_{\bar{\zeta}}$. The signs agree with [20, (C.1)–(C.4)].

Differentiating (1.1) and using (3.6) gives

$$\begin{aligned} (Q_{\text{AR}})_{\bar{\zeta}} &= 2H_{\bar{\zeta}}p + 2(H + i\tau)p_{\bar{\zeta}} - 2kA A_{\bar{\zeta}} \\ &= 2H_{\bar{\zeta}}p + \lambda(H + i\tau)H_{\zeta} + ku\lambda A(H + i\tau) - ku\lambda A(H + i\tau). \end{aligned}$$

The terms $ku\lambda A(H + i\tau)$ cancel, leaving, for every immersion,

$$(3.9) \quad (Q_{\text{AR}})_{\bar{\zeta}} = 2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\zeta}.$$

Thus holomorphy is equivalent to

$$(3.10) \quad 2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\zeta} = 0.$$

If H is constant, both derivatives vanish, proving the CMC-to-holomorphy implication in Theorem 1.1. The identity (3.9), valid for every immersion, is the starting point for the converse: on the open set where $dH \neq 0$, equation (3.10) determines the Hopf coefficient p in terms of H and λ , and this rigidly fixes the shape operator (Proposition 3.3 below).

Proposition 3.1 (Comparison of the holomorphy conditions). *For any immersion with $\tau \neq 0$, with no CMC hypothesis,*

$$(Q_{\text{FM}})_{\bar{\zeta}} = \lambda H_{\zeta} + \frac{kA^2}{(H + i\tau)^2} H_{\bar{\zeta}}, \quad (Q_{\text{AR}})_{\bar{\zeta}} = (H + i\tau)(Q_{\text{FM}})_{\bar{\zeta}} + H_{\bar{\zeta}} Q_{\text{FM}}.$$

Consequently the holomorphy equations of Q_{AR} and Q_{FM} are, respectively,

$$2pH_{\bar{\zeta}} + \lambda(H + i\tau)H_{\zeta} = 0, \quad kA^2 H_{\bar{\zeta}} + \lambda(H + i\tau)^2 H_{\zeta} = 0,$$

and $Q_{\text{AR}}, Q_{\text{FM}}$ have the same zero set.

Proof. Differentiating (2.1),

$$(Q_{\text{FM}})_{\bar{\zeta}} = 2p_{\bar{\zeta}} - k \left[\frac{2AA_{\bar{\zeta}}}{H + i\tau} - \frac{A^2 H_{\bar{\zeta}}}{(H + i\tau)^2} \right].$$

By (3.6), $2p_{\bar{\zeta}} = \lambda H_{\bar{\zeta}} + ku\lambda A$ and

$$\frac{2kAA_{\bar{\zeta}}}{H + i\tau} = \frac{2kA}{H + i\tau} \cdot \frac{u\lambda}{2}(H + i\tau) = ku\lambda A.$$

The two terms $ku\lambda A$ cancel, which gives the first identity. The second is the Leibniz rule applied to $Q_{\text{AR}} = (H + i\tau)Q_{\text{FM}}$, using $(H + i\tau)_{\bar{\zeta}} = H_{\bar{\zeta}}$. The holomorphy equation of Q_{AR} is (3.10); that of Q_{FM} follows from the first identity after multiplication by $(H + i\tau)^2 \neq 0$ (H real, $\tau \neq 0$). Finally $H + i\tau \neq 0$ everywhere, so $Q_{\text{AR}} = 0 \iff Q_{\text{FM}} = 0$. $\square$

Equality of the zero sets does not make the two holomorphy conditions equivalent: if Q_{FM} is holomorphic, then $(Q_{\text{AR}})_{\bar{\zeta}} = H_{\bar{\zeta}}Q_{\text{FM}}$. Fernández-Mira [20, Remark 2.6, p. 285] point out this distinction (they write $P d\zeta^2 = (H + i\tau)Q d\zeta^2$ for the original differential); by Theorem 1.1, their exceptional surfaces in $\widetilde{PSL}_2(\mathbb{R})$ [20, Theorem 1.1] do not have holomorphic Q_{AR} .

Remark 3.2 (Hopf cylinders with holomorphic Q_{AR} are CMC). Independently of Theorem 1.1: a Hopf cylinder $\pi^{-1}(\gamma)$ is characterized by $u \equiv 0$, so $T = \xi$ is parallel by (3.3); with the parallel frame $e_2 = \xi$, $e_1 = -Je_2$ and flat coordinates ($\partial_x = e_1$, $\partial_y = e_2$, $\lambda = 1$), (3.4) gives $S_{22} = 0$, $S_{12} = \tau$, hence $p = (H - i\tau)/2$, $A = -i/2$ and

$$Q_{\text{AR}} = (H + i\tau)(H - i\tau) + \frac{k}{4} = H^2 + \frac{\kappa}{4}.$$

Since H is constant along the fibers, every datum is a function of x alone, and a holomorphic function of x alone is constant; so holomorphy forces H^2 , hence H , to be constant. The same argument applies to the normalized differentials, e.g. $Q_{\text{AR}}/(2H + i) = H^2/(2H + i)$ in Nil_3 .

Assume from now on that Q_{AR} is holomorphic and that $\Omega = \{dH \neq 0\}$ is nonempty; write $f = |\nabla H| > 0$, $e_1 = \nabla H/f$, $e_2 = Je_1$, a globally defined smooth positively oriented orthonormal frame on a fixed connected component of Ω , which we continue to denote by Ω .

Proposition 3.3 (Normal form on the non-CMC set). *If Q_{AR} is holomorphic, then on Ω*

$$(3.11) \quad [S]_{(e_1, e_2)} = \begin{pmatrix} 0 & \tau \\ \tau & 2H \end{pmatrix}, \quad \det S = -\tau^2, \quad K = ku^2.$$

In particular, ∇H is an asymptotic direction at every point of Ω , and the principal curvatures are $H \pm \sqrt{H^2 + \tau^2}$.

Proof. At $x_0 \in \Omega$, rotate the conformal parameter, $\zeta \mapsto e^{i\theta_0}\zeta$, so that $H_\zeta(x_0) > 0$; then λ is unchanged, (3.10) is invariant (both terms acquire the factor $e^{-i\theta_0}$), and at x_0 the frame $(\partial_x, \partial_y)/\sqrt{\lambda}$ is (e_1, e_2) , so $H_\zeta = H_{\bar{\zeta}} = f\sqrt{\lambda}/2$. Then (3.10) gives $p(x_0) = -\lambda(x_0)(H(x_0) + i\tau)/2$. Since $p = \frac{\lambda}{4}(S_{11} - S_{22} - 2iS_{12})$ (from $\partial_\zeta = \frac{\sqrt{\lambda}}{2}(e_1 - ie_2)$), this gives $S_{11} - S_{22} = -2H$ and $S_{12} = \tau$; with $S_{11} + S_{22} = 2H$, $S_{11} = 0$ and $S_{22} = 2H$. Then $\det S = -\tau^2$, and (3.1) gives $K = \det S + \tau^2 + ku^2 = ku^2$. Finally $\langle Se_1, e_1 \rangle = S_{11} = 0$, so $e_1 = \nabla H/|\nabla H|$ is asymptotic, and the eigenvalues of S are $H \pm \sqrt{H^2 - \det S} = H \pm \sqrt{H^2 + \tau^2}$. $\square$

The normal form is a pointwise consequence of (3.10) and (3.1); no analyticity or global hypothesis is used. Thus every point of Ω is hyperbolic, with $\kappa_1\kappa_2 = -\tau^2$, and ∇H is asymptotic but not principal ($\langle Se_1, e_2 \rangle = \tau \neq 0$), in contrast with the product case $\tau = 0$, where ∇H is principal with principal curvature 0 [8, pp. 172–173]. We now show that the structure equations are incompatible with this normal form.

On Ω write $\nabla_{e_1}e_1 = a e_2$ and $\nabla_{e_2}e_1 = b e_2$, so that a, b are the coefficients of the Levi-Civita connection in the frame (e_1, e_2) , and put $\rho^2 = H^2 + \tau^2 > 0$. On the open subset of Ω where $R := \sqrt{1 - u^2} > 0$, choose locally a smooth angle function θ with

$$T = \xi - uN = R(\cos \theta e_1 + \sin \theta e_2),$$

and put

$$P = \tau \cos \theta + H \sin \theta, \quad L = -\tau \sin \theta + H \cos \theta,$$

so that $P^2 + L^2 = \rho^2$.

Lemma 3.4 (Frame equations). *If $\mathcal{Q}_{\text{AR}}$ is holomorphic, then $e_1u = 0$ on Ω , and, where $R > 0$,*

$$(3.12) \quad a = -\frac{\tau f}{\rho^2} - \frac{kuRP}{2\rho^2}, \quad b = -\frac{Hf}{\rho^2} - \frac{kuRL}{2\rho^2},$$

$$(3.13) \quad e_1u = 0, \quad e_2u = -2RP,$$

$$(3.14) \quad e_1\theta = -a, \quad e_2\theta = -b + \frac{2uL}{R}, \quad e_2f = af.$$

Proof. We use the normal form (3.11) throughout.

Step 1: $e_1u = 0$ and $e_2u = -2RP$. Since $Se_1 = \tau e_2$, $Se_2 = \tau e_1 + 2He_2$, $Je_1 = e_2$ and $Je_2 = -e_1$,

$$(3.15) \quad (S - \tau J)e_1 = 0, \quad (S - \tau J)e_2 = 2(\tau e_1 + He_2).$$

By (3.4), $e_1u = -\langle (S - \tau J)e_1, T \rangle = 0$. Where $R > 0$, write $T = R(\cos \theta e_1 + \sin \theta e_2)$; then $e_2u = -\langle 2\tau e_1 + 2He_2, T \rangle = -2R(\tau \cos \theta + H \sin \theta) = -2RP$.

Step 2: the connection coefficients. Write $\nabla_{e_1}e_1 = ae_2$, $\nabla_{e_1}e_2 = -ae_1$, $\nabla_{e_2}e_1 = be_2$, $\nabla_{e_2}e_2 = -be_1$. Using $(\nabla_X S)Y = \nabla_X(SY) - S\nabla_X Y$ and

$e_1H = f$, $e_2H = 0$, we obtain

$$\begin{aligned} (\nabla_{e_1}S)e_2 &= \nabla_{e_1}(\tau e_1 + 2He_2) - S(-ae_1) \\ &= \tau ae_2 + 2fe_2 - 2Hae_1 + a\tau e_2 = -2Ha e_1 + (2\tau a + 2f)e_2, \\ (\nabla_{e_2}S)e_1 &= \nabla_{e_2}(\tau e_2) - S(be_2) \\ &= -\tau b e_1 - b(\tau e_1 + 2He_2) = -2\tau b e_1 - 2Hb e_2. \end{aligned}$$

Subtracting and comparing with the right side $kuR \sin \theta e_1 - kuR \cos \theta e_2$ of (3.2) gives $-2Ha + 2\tau b = kuR \sin \theta$ and $2\tau a + 2Hb = -2f - kuR \cos \theta$, a linear system of determinant $-4\rho^2 \neq 0$ whose solution is (3.12).

Step 3: the angular equations. Since $e_1u = 0$, $e_1R = 0$, and $\nabla_{e_1}T = R(e_1\theta + a)(-\sin \theta e_1 + \cos \theta e_2)$; by (3.3) and (3.15), $\nabla_{e_1}T = u(S - \tau J)e_1 = 0$, so $e_1\theta = -a$. Similarly $e_2R = 2uP$, and the component of $\nabla_{e_2}T$ along $-\sin \theta e_1 + \cos \theta e_2$ is $R(e_2\theta + b)$, while $u(S - \tau J)e_2 = 2u(\tau e_1 + He_2)$ has component $2uL$; hence $e_2\theta = -b + 2uL/R$.

Step 4: $e_2f = af$. Since $f = e_1H$, $e_2f = e_2e_1H$. By torsion-freeness, $[e_2, e_1] = \nabla_{e_2}e_1 - \nabla_{e_1}e_2 = be_2 + ae_1$, so $e_2e_1H - e_1e_2H = [e_2, e_1]H = af$. Since $e_2H \equiv 0$ on Ω , $e_1e_2H = 0$, and hence $e_2f = e_2e_1H = af$. $\square$

Put $\Omega' := \Omega \cap \{|u| < 1\}$, where $R > 0$, and

$$\Omega_* := \{x \in \Omega' : H(x) \neq 0, u(x) \neq 0, P(x) \neq 0\}.$$

On Ω' the unit vector T/R , and with it $\cos \theta$, $\sin \theta$, P and L , are well defined and smooth, so Ω_* is open.

Lemma 3.5 (Regular coordinate region). *If $\mathcal{Q}_{AR}$ is holomorphic, then Ω_* is dense in Ω , and (H, u) are local coordinates on Ω_* .*

Proof. Since the complement of a closed set with empty interior is open and dense, it suffices to show that $\{|u| = 1\}$ has empty interior in Ω and that the relatively closed sets $\{H = 0\}$, $\{u = 0\}$ and $\{P = 0\}$ of Ω' have empty interior in Ω' .

Step 1: $\{|u| = 1\}$. If $|u| \equiv 1$ on a nonempty open $U \subset \Omega$, then $|T|^2 = 1 - u^2 \equiv 0$, so $\nabla T \equiv 0$ on U . By (3.3) and (3.15), $\nabla_{e_2}T = 2u(\tau e_1 + He_2) \neq 0$, since $u = \pm 1$ and $\tau \neq 0$, a contradiction.

Step 2: $\{H = 0\}$. If $H \equiv 0$ on an open $U \subset \Omega'$, then $dH = 0$ there, contradicting $U \subset \Omega$.

Step 3: $\{P = 0\}$. Suppose $P \equiv 0$ on a nonempty open $U \subset \Omega'$; by Step 2 we may shrink U so that $H \neq 0$ on U . We derive a contradiction in three substeps.

(i) *u and θ are constant along e_2 .* By (3.13), $e_2u = -2RP = 0$. Since $e_2H = 0$,

$$0 = e_2P = (-\tau \sin \theta + H \cos \theta) e_2\theta = L e_2\theta,$$

and $L \neq 0$ because $P^2 + L^2 = \rho^2 > 0$; hence $e_2\theta = 0$.

(ii) f is a function of H , u and θ . By (i) and (3.14), $b = 2uL/R$. Comparing with (3.12) and solving for f ,

$$\frac{2uL}{R} = -\frac{Hf}{\rho^2} - \frac{kuRL}{2\rho^2} \implies f = -\frac{uL(4\rho^2 + kR^2)}{2HR}.$$

(iii) *Contradiction.* Every factor on the right of (ii) has vanishing e_2 -derivative: $e_2u = 0$, $e_2\rho^2 = 2He_2H = 0$, $e_2R = -ue_2u/R = 0$, and $e_2L = (e_2H)\cos\theta - Pe_2\theta = 0$. Hence $e_2f = 0$. On the other hand, (3.14) and (3.12) with $P = 0$ give $e_2f = af = -\tau f^2/\rho^2 \neq 0$.

Step 4: $\{u = 0\}$. If $u \equiv 0$ on a nonempty open $U \subset \Omega'$, then $R = 1$ and (3.13) gives $P \equiv 0$ on U , which is excluded by Step 3.

Hence Ω_* is dense in Ω . Finally, $(dH \wedge du)(e_1, e_2) = f \cdot (-2RP) - 0 = -2fRP \neq 0$ on Ω_* , and the inverse function theorem shows that (H, u) are local coordinates there. $\square$

Under the homothety $\bar{g} \mapsto \mathfrak{c}^2\bar{g}$, $\mathfrak{c} > 0$,

$$(\kappa, \tau, H, \lambda, p, A) \longmapsto (\kappa/\mathfrak{c}^2, \tau/\mathfrak{c}, H/\mathfrak{c}, \mathfrak{c}^2\lambda, \mathfrak{c}p, \mathfrak{c}A).$$

Indeed, $S \mapsto S/\mathfrak{c}$, so $p \mapsto \mathfrak{c}^2p/\mathfrak{c} = \mathfrak{c}p$; the vertical component scales by $\mathfrak{c}$, while u is unchanged. The ambient sectional curvatures scale by $\mathfrak{c}^{-2}$. Consequently,

$$2\left(\frac{H}{\mathfrak{c}} + i\frac{\tau}{\mathfrak{c}}\right)(\mathfrak{c}p) - \left(\frac{\kappa - 4\tau^2}{\mathfrak{c}^2}\right)(\mathfrak{c}A)^2 = 2(H + i\tau)p - kA^2 = Q_{\text{AR}}.$$

Take $\mathfrak{c} = |\tau|$. If $\tau < 0$, recall that reversing the orientation of $E(\kappa, \tau)$ gives an orientation-reversing isometry $E(\kappa, \tau) \rightarrow E(\kappa, -\tau)$ (the sign of τ in $\bar{\nabla}_X\xi = \tau X \times \xi$ depends on the orientation through the cross product). Reverse accordingly the ambient and induced surface orientations, keeping N, ξ . This reverses τ , replaces ζ by $\bar{\zeta}$, and conjugates p, A, Q_{AR} , preserving holomorphy. For $k \neq 0$ we may therefore assume

$$(3.16) \quad \tau = 1, \quad q = \frac{\kappa - 4\tau^2}{\tau^2} \neq 0.$$

The parameter $q = k/\tau^2$ is invariant under both operations.

3.1. A connection adapted to the fibration. We close this section with a geometric reading of the normal form, which explains the role of k and the dichotomy between Sections 4 and 5. It is not used in the proof of Theorem 1.1. Orient $E(\kappa, \tau)$ so that $\bar{\nabla}_X\xi = \tau X \times \xi$, and define

$$(3.17) \quad D_XY := \bar{\nabla}_XY - \tau X \times Y.$$

Since $Y \mapsto X \times Y$ is skew-adjoint, D is a metric connection; its torsion $-2\tau X \times Y$ is totally skew-symmetric, and $D_X\xi = \tau X \times \xi - \tau X \times \xi = 0$. Thus ξ is D -parallel, and D preserves the horizontal distribution $\xi^\perp$. Moreover D commutes with $Z \mapsto \xi \times Z$: expanding $D_X(\xi \times Y) - \xi \times D_XY$ with $\bar{\nabla}_X\xi = \tau X \times \xi$ leaves $\tau[(X \times \xi) \times Y + \xi \times (X \times Y) + X \times (Y \times \xi)]$, which vanishes by the Jacobi identity.

Proposition 3.6 (Curvature of D). *The curvature of D is*

$$R^D(X, Y)Z = -k \langle X \times Y, \xi \rangle \xi \times Z.$$

In particular D is flat if and only if $k = 0$, and its holonomy consists of rotations about ξ .

Proof. Write $D = \bar{\nabla} + \mathcal{T}$ with $\mathcal{T}_X Y = -\tau X \times Y$. The cross product is $\bar{\nabla}$ -parallel, so $\bar{\nabla} \mathcal{T} = 0$ and $R^D(X, Y) = \bar{R}(X, Y) + [\mathcal{T}_X, \mathcal{T}_Y]$. By the Jacobi identity, $[\mathcal{T}_X, \mathcal{T}_Y]Z = \tau^2(X \times Y) \times Z = -\tau^2(\langle Y, Z \rangle X - \langle X, Z \rangle Y)$. Translated to the present sign convention, the curvature of $E(\kappa, \tau)$ is [11, Proposition 2.1]

$$\begin{aligned} \bar{R}(X, Y)Z &= (\kappa - 3\tau^2)(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &\quad - k(\langle Y, \xi \rangle \langle Z, \xi \rangle X - \langle X, \xi \rangle \langle Z, \xi \rangle Y \\ &\quad + \langle X, \xi \rangle \langle Y, Z \rangle \xi - \langle Y, \xi \rangle \langle X, Z \rangle \xi). \end{aligned}$$

Hence $R^D(X, Y)Z = k\mathcal{C}(X, Y)Z$, where $\mathcal{C}(X, Y)Z$ is $\langle Y, Z \rangle X - \langle X, Z \rangle Y$ minus the bracket above. A direct check shows $\mathcal{C}(X, Y)Z = 0$ whenever one of X, Y, Z equals ξ , so $\mathcal{C}$ only sees horizontal components; on a positively oriented horizontal orthonormal pair E_1, E_2 (so $E_1 \times E_2 = \xi$, $\xi \times E_1 = E_2$, $\xi \times E_2 = -E_1$) one has $\mathcal{C}(E_1, E_2)E_1 = -E_2 = -\xi \times E_1$ and $\mathcal{C}(E_1, E_2)E_2 = E_1 = -\xi \times E_2$. Both sides of the claimed formula are trilinear, which proves it. $\square$

For any immersion, $JX = N \times X$ gives

$$(3.18) \quad D_X N = -(S - \tau J)X, \quad \det(S - \tau J) = \det S + \tau^2 = K - ku^2.$$

Proposition 3.7. *On $\Omega = \{dH \neq 0\}$, $\mathcal{Q}_{\text{AR}}$ is holomorphic if and only if $D_{\nabla H} N = 0$.*

Proof. By (3.18), $D_{e_1} N = 0$ if and only if $Se_1 = \tau e_2$, i.e. $S_{11} = 0$, $S_{12} = \tau$, which is the normal form (3.11); in the rotated parameter of the proof of Proposition 3.3 this is $p = -\lambda(H + i\tau)/2$, i.e. (3.10). $\square$

On $\Omega' = \Omega \cap \{|u| < 1\}$ let $\nu = (N - u\xi)/R$ be the normalized horizontal part of N and $\sigma(X) = \langle D_X \nu, \xi \times \nu \rangle$. Since D preserves $\xi^\perp$ and commutes with $\xi \times$, Proposition 3.6 gives $d\sigma(X, Y) = -k\langle X \times Y, \xi \rangle$, and with $\omega = ae_1^\flat + be_2^\flat$ the connection form of (e_1, e_2) , Lemma 3.4 reads

$$(3.19) \quad d\sigma = -ku dA, \quad du = -2RP e_2^\flat, \quad \sigma = \frac{2L}{R} e_2^\flat, \quad d\theta = u\sigma - \omega.$$

The integrability condition $d(d\theta) = 0$ is the Gauss equation $K = ku^2$. For $k = 0$, σ is closed, which is why the round case integrates directly (Section 4). For $k \neq 0$ it is not; the curvature of D alone does not exclude non-CMC surfaces, and the obstruction comes from the full system of Section 5.

4. PROOF OF THEOREM 1.1 FOR $k = 0$

In the round case $k = 0$ the original differential is $Q_{\text{AR}} = 2(H + i\tau)p$, and the algebraic reduction of Section 5 is not needed: the proof uses only (3.10), the Gauss equation and Liouville's formula.

Proof of Theorem 1.1 for $k = 0$. Suppose Q_{AR} is holomorphic and, for contradiction, that $\Omega = \{dH \neq 0\} \neq \emptyset$. Fix a point of Ω and a simply connected coordinate disc $\mathcal{D} \subset \Omega$ around it, with conformal parameter $\zeta = x + iy$, $g = \lambda|d\zeta|^2$. Put $\rho^2 := H^2 + \tau^2 > 0$.

Step 1: a natural parameter. Taking moduli in (3.10), $|p| = \lambda\rho/2 > 0$, so $Q_{\text{AR}} = 2(H + i\tau)p$ has no zeros on $\mathcal{D}$. Choose a holomorphic ω with $e^\omega = Q_{\text{AR}}$ [3, Ch. 4] and the new conformal parameter $\eta = \int e^{\omega/2} d\zeta$, so that $d\eta^2 = Q_{\text{AR}}$; the relation $|p| = \lambda\rho/2$ is invariant, since $|p|$ and λ acquire the same factor. In this parameter $Q_{\text{AR}} \equiv 1$, so

$$(4.1) \quad p = \frac{1}{2(H + i\tau)}, \quad \lambda = \frac{2|p|}{\rho} = \frac{1}{\rho^2}.$$

Step 2: the equation for H . Substituting (4.1) in (3.10),

$$\frac{H_{\bar{\zeta}}}{H + i\tau} + \frac{(H + i\tau)H_{\zeta}}{\rho^2} = 0.$$

Multiplying by $\rho^2 = (H + i\tau)(H - i\tau)$ gives $(H - i\tau)H_{\bar{\zeta}} + (H + i\tau)H_{\zeta} = 0$. As $H_{\bar{\zeta}} = \overline{H_{\zeta}}$, this says $2\operatorname{Re}[(H + i\tau)H_{\zeta}] = 0$, and with $H_{\zeta} = \frac{1}{2}(H_x - iH_y)$,

$$(4.2) \quad HH_x + \tau H_y = 0.$$

The Codazzi equation $p_{\bar{\zeta}} = \lambda H_{\zeta}/2$ gives the same equation.

Step 3: Gauss. By (3.5) and (4.1), $\det S = H^2 - 4|p|^2/\lambda^2 = H^2 - \rho^2 = -\tau^2$, so (3.1) gives $K = 0$. By Liouville's formula $\Delta_0 \log \lambda = 0$, where $\Delta_0 = \partial_x^2 + \partial_y^2$ is the coordinate Laplacian of (3.5); since $\log \lambda = -\log \rho^2$,

$$(4.3) \quad \Delta_0 \psi = 0, \quad \psi := \log(H^2 + \tau^2).$$

Step 4: computation. Put $\beta := H_x$. By (4.2), $H_y = -H\beta/\tau$. Differentiating (4.2) in x ,

$$(4.4) \quad \tau\beta_y + \beta^2 + H\beta_x = 0.$$

Differentiating $H_y = -H\beta/\tau$ in y and using (4.4),

$$H_{yy} = -\frac{H_y\beta + H\beta_y}{\tau} = \frac{H\beta^2}{\tau^2} - \frac{H\beta_y}{\tau} = \frac{H}{\tau^2}(2\beta^2 + H\beta_x).$$

Moreover, for the coordinate gradient $\nabla_0 H := (H_x, H_y)$, $|\nabla_0 H|^2 = \beta^2 + H^2\beta^2/\tau^2 = \beta^2\rho^2/\tau^2$. Since ψ is a function of H with $\psi' = 2H/\rho^2$ and $\psi'' = 2(\tau^2 - H^2)/\rho^4$, the chain rule $\Delta_0 \psi = \psi''|\nabla_0 H|^2 + \psi' \Delta_0 H$, $\Delta_0 H = \beta_x + H_{yy}$,

gives

$$\begin{aligned}\Delta_0\psi &= \frac{2(\tau^2 - H^2)\beta^2}{\tau^2\rho^2} + \frac{2H}{\rho^2} \left[\beta_x + \frac{H}{\tau^2}(2\beta^2 + H\beta_x) \right] \\ &= \frac{2\rho^2\beta^2 + 2H\rho^2\beta_x}{\tau^2\rho^2} = \frac{2}{\tau^2}(\beta^2 + H\beta_x),\end{aligned}$$

because $2(\tau^2 - H^2)\beta^2 + 4H^2\beta^2 = 2\rho^2\beta^2$ and $2H\tau^2\beta_x + 2H^3\beta_x = 2H\rho^2\beta_x$. (Equivalently, $\Delta_0\psi = (H^2)_{xx}/\tau^2$.) By (4.3),

$$(4.5) \quad \beta^2 + H\beta_x \equiv 0 \quad \text{on } \mathcal{D}, \quad \text{and then (4.4) gives } \beta_y \equiv 0.$$

Step 5: contradiction. If $\beta(x_0) = 0$ at some $x_0 \in \mathcal{D}$, then $H_y(x_0) = -H\beta/\tau = 0$, so $dH(x_0) = 0$, contradicting $\mathcal{D} \subset \Omega$. Hence $\beta \neq 0$ on $\mathcal{D}$. By (4.5), $H = 0$ would force $\beta^2 = 0$; so $H \neq 0$ and $H_y = -H\beta/\tau \neq 0$ on $\mathcal{D}$. Differentiating $\beta^2 + H\beta_x = 0$ in y and using $\beta_y = \beta_{xy} = 0$ gives $H_y\beta_x = 0$, hence $\beta_x = 0$ and then $\beta^2 = 0$, a contradiction.

Thus Ω is empty and H is constant. The converse follows from (3.9). $\square$

5. PROOF OF THEOREM 1.1 FOR $k \neq 0$

Remark 5.1 (Why the round argument does not extend). For $k = 0$ the normalization $Q_{\text{AR}} \equiv 1$ fixes p and λ as functions of H . For $k \neq 0$ the term $-kA^2$, with $|A|^2 = \lambda(1 - u^2)/4$, brings in u as a second unknown, Q_{AR} may vanish on Ω , and the normalization determines neither p nor λ ; equivalently, $d\sigma = -ku dA \neq 0$ in (3.19). We therefore work with the coordinates (H, u) .

We first outline the argument for $k \neq 0$. Suppose that Q_{AR} is holomorphic and $\Omega \neq \emptyset$. By Lemma 3.5 we may work near a point of Ω_* , where (H, u) are coordinates.

- (i) In §5.1 the frame equations of Lemma 3.4 become a first-order rational system (5.3)–(5.5), (5.8) for two functions z, w of (H, u) .
- (ii) Equality of the mixed derivatives of w yields a polynomial relation $\Pi(H, u, w, z) = 0$, and hence also $X\Pi = 0$ (Lemma 5.3).
- (iii) In §5.2: every smooth solution of the system is real analytic (Lemma 5.4; analyticity is a consequence, not a hypothesis), so z and w generate, over $F := \mathbb{C}(H, u)$, a field E of germs.
- (iv) Eliminating z between Π and $X\Pi$ by a resultant gives a nontrivial polynomial equation for w over F (Lemmas 5.6 and 5.7); hence E is a finite extension of F .
- (v) In §5.3 we embed E in a field of Puiseux series in u , preserving ∂_H and ∂_u (Lemma 5.8), and show by comparing orders at $u = 0$ that the system has no solution there (Lemma 5.10).
- (vi) In §5.4 these steps are combined to prove Theorem 1.1 for $k \neq 0$.

Fix $q \neq 0$ and work on Ω_* , where $H \neq 0$, $0 < |u| < 1$, $P \neq 0$, and (H, u) are coordinates. Put

$$s = 1 + H^2, \quad \ell = 4s + q, \quad C = \frac{1}{1 - u^2} + \frac{q}{4s}.$$

On a simply connected neighborhood choose the angle θ and set

$$(5.1) \quad \alpha := \arctan H - \theta, \quad z := \tan \alpha, \quad w := s \alpha_H.$$

With $\varphi = \arctan H$, so that $\cos \varphi = 1/\sqrt{s}$ and $\sin \varphi = H/\sqrt{s}$, the addition formulas and the definitions of P, L (with $\tau = 1$) give

$$(5.2) \quad P = \sqrt{s} \cos \alpha, \quad L = \sqrt{s} \sin \alpha, \quad z = \tan \alpha = \frac{L}{P},$$

a real function, since $P \neq 0$ on Ω_* .

On Ω_* we have $H \neq 0$, $0 < |u| < 1$, $q \neq 0$, $1 + z^2 \geq 1$, and $w \neq 0$ by (5.6) below; thus (H, u, z, w) lies in $V := \{Huw(1 - u^2) \neq 0\}$, and all divisions below are legitimate.

5.1. The rational system and a polynomial relation. The angular equations give

$$(5.3) \quad z_H = (1 + z^2) \frac{w}{s},$$

$$(5.4) \quad z_u = (1 + z^2) \mathcal{A}, \quad \mathcal{A} := -\frac{qHu}{4sw} + uzC,$$

$$(5.5) \quad w_u = \mathcal{G}, \quad \mathcal{G} := \frac{w}{u} + \frac{qu(Hz + 1)}{4s} - uwC(1 + z^2).$$

Indeed, $e_1 = f\partial_H$ and $e_2 = -2RP\partial_u$, as both sides agree on H and u by (3.13). By (3.14) and (3.12) (with $\tau = 1$, $k = q$), $e_1\alpha = f/s + a = -quRP/(2s)$, so $\alpha_H = -quRP/(2sf)$ and

$$(5.6) \quad w = s\alpha_H = -\frac{quRP}{2f}, \quad \text{i.e.} \quad \frac{f}{RP} = -\frac{qu}{2w};$$

then $z_H = (1 + z^2)\alpha_H$ is (5.3). Next, $\alpha_u = e_2\alpha/(-2RP) = (-b + 2uL/R)/(2RP)$; substituting b from (3.12), (5.6), $L/P = z$ and $R^2 = 1 - u^2$ gives $\alpha_u = \mathcal{A}$, hence (5.4). Finally, $w = -quR\sqrt{s}\cos\alpha/(2f)$ by (5.2), so $w_u/w = 1/u - u/(1 - u^2) - z\alpha_u - f_u/f$; since $e_2f = af$, $f_u/f = -a/(2RP) = -qu/(4sw) + qu/(4s)$, and substituting gives (5.5).

To determine w_H , use $(\alpha_u)_H = (\alpha_H)_u$. Introduce

$$(5.7) \quad X := \partial_H + (1 + z^2) \frac{w}{s} \partial_z + \mathcal{B} \partial_w, \quad Y := \partial_u + (1 + z^2) \mathcal{A} \partial_z + \mathcal{G} \partial_w$$

where $\mathcal{B} = w_H$. Along a solution, these vector fields act as the total derivatives with respect to H, u .

Differentiating $\alpha_u = \mathcal{A}$ along the graph gives

$$(\alpha_u)_H = \mathcal{A}_H + \mathcal{A}_z z_H + \mathcal{A}_w w_H = \mathcal{A}_H + \mathcal{A}_z (1 + z^2) \frac{w}{s} + \mathcal{A}_w \mathcal{B}.$$

Since $(\alpha_H)_u = (w/s)_u = \mathcal{G}/s$, equality of mixed derivatives gives

$$(5.8) \quad w_H = \mathcal{B} := \frac{\mathcal{G}/s - \mathcal{A}_H - \mathcal{A}_z (1 + z^2) w/s}{\mathcal{A}_w},$$

$$\mathcal{A}_w = \frac{qHu}{4sw^2} \neq 0.$$

Here subscripts on $\mathcal{A}$ are formal partial derivatives. With $C_H = -qH/(2s^2)$, $\mathcal{A}_z = uC$ and $\mathcal{A}_H = -qu(1 - H^2)/(4s^2w) + uzC_H$, a direct simplification of (5.8), dividing term by term by $\mathcal{A}_w$, gives

$$(5.9) \quad \mathcal{B} = \frac{4w^3}{qHu^2} + \frac{w^2(3Hz + 1) + w(1 - H^2)}{Hs} - \frac{8w^3C(1 + z^2)}{qH}.$$

From now on, X and Y denote the vector fields (5.7) with $\mathcal{B}$ the explicit rational function (5.9).

Remark 5.2 (Geometric meaning of the system). With $\tau = 1$, $\sigma = -z du/(1 - u^2)$, and (5.3)–(5.4) are the two components of $d\theta = u\sigma - \omega$ in (3.19), i.e. of the transport law (3.3); (5.5) is the symmetry $e_2f = af$ of the Hessian of H ; and (5.8) comes from $d(d\theta) = 0$, i.e. the Gauss equation. Thus $\Pi = 0$ below expresses the compatibility of the Gauss equation with the symmetry of the Hessian of H for a surface whose normal is D -parallel along ∇H .

We now derive a polynomial relation and use it to prove that w, z are algebraic over $\mathbb{C}(H, u)$.

Define the polynomials

$$\delta := 6s\ell - qu^2(14s + q) + q^2u^4$$

and

$$(5.10) \quad \begin{aligned} \Pi := & 4\delta w^3(1 + z^2) \\ & + q[7u^2(\ell - qu^2)z^2 + 4s(11u^2 - 4) + 7qu^2(1 - u^2)]w^2 \\ & - 8Hq(1 - u^2)w(3sw + qu^2)z \\ & - 2q^2u^2(1 - u^2)(1 - 3H^2)w - q^2su^2(1 - u^2) \end{aligned}$$

in $\mathbb{C}[H, u, w, z]$.

Lemma 5.3 (Polynomial compatibility). *Every solution of (5.3)–(5.8) satisfies $\Pi = X\Pi = 0$.*

Proof. On the graph $(H, u, z(H, u), w(H, u))$, X, Y are the pushforwards of ∂_H, ∂_u . Hence $[X, Y]$ vanishes there.

The ∂_H - and ∂_u -components of $[X, Y]$ vanish identically. We calculate the other two components.

Put $\Xi = (1 + z^2)w/s$. The ∂_z -component is

$$X((1 + z^2)\mathcal{A}) - Y\Xi = (1 + z^2)\left(\mathcal{A}_H + \Xi\mathcal{A}_z + \mathcal{B}\mathcal{A}_w - \frac{\mathcal{G}}{s}\right) + 2z\Xi\mathcal{A} - 2z\Xi\mathcal{A}.$$

The terms $2z\Xi\mathcal{A}$ cancel, and the remaining bracket is zero by (5.8). Thus this component vanishes identically.

For the ∂_w -component, write $\mathcal{A} = a_{-1}w^{-1} + a_0$, $\mathcal{G} = g_1w + g_0$, and $\mathcal{B} = b_3w^3 + b_2w^2 + b_1w$, where

$$\begin{aligned} a_{-1} &= -\frac{qHu}{4s}, & a_0 &= uzC, \\ g_1 &= \frac{1}{u} - uC(1+z^2), & g_0 &= \frac{qu(Hz+1)}{4s}, \\ b_3 &= \frac{4}{qHu^2} - \frac{8C(1+z^2)}{qH}, & b_2 &= \frac{3Hz+1}{Hs}, & b_1 &= \frac{1-H^2}{Hs}. \end{aligned}$$

Put $v = (1+z^2)/s$ and $\tilde{a}_j = (1+z^2)a_j$ for $j = -1, 0$. No negative powers of w occur in $X\mathcal{G} - Y\mathcal{B}$: the only possible pole, in $\mathcal{A}$, is canceled by the factor w in $\partial_z\mathcal{B}$.

Expanding $X\mathcal{G} = \partial_H\mathcal{G} + vw\partial_z\mathcal{G} + \mathcal{B}\partial_w\mathcal{G}$ and $Y\mathcal{B} = \partial_u\mathcal{B} + (\tilde{a}_{-1}w^{-1} + \tilde{a}_0)\partial_z\mathcal{B} + \mathcal{G}\partial_w\mathcal{B}$ in powers of w , one finds $X\mathcal{G} - Y\mathcal{B} = \sum_{j=0}^3 r_j w^j$, where

$$\begin{aligned} (5.11) \quad r_3 &= -(b_3)_u - (1+z^2)a_0(b_3)_z - 2g_1b_3, \\ r_2 &= v(g_1)_z - (b_2)_u - (1+z^2)(a_{-1}(b_3)_z + a_0(b_2)_z) \\ &\quad - g_1b_2 - 3g_0b_3, \\ r_1 &= (g_1)_H + v(g_0)_z - (b_1)_u - (1+z^2)(a_{-1}(b_2)_z + a_0(b_1)_z) \\ &\quad - 2g_0b_2, \\ r_0 &= (g_0)_H - (1+z^2)a_{-1}(b_1)_z - g_0b_1. \end{aligned}$$

Substituting the explicit rational functions $a_{-1}, a_0, g_1, g_0, b_3, b_2, b_1$ recorded above into the four formulas (5.11) and simplifying—a direct but lengthy application of the product and quotient rules to functions already on hand, carried out in full in Fact A.1—gives

$$\begin{aligned} r_0 &= -\frac{qu}{4Hs}, & r_1 &= -\frac{qu(1-3H^2+4Hz)}{2Hs^2}, \\ r_2 &= -\frac{J_2}{4Hus^2(u^2-1)}, & r_3 &= -\frac{\delta(1+z^2)}{qHus^2(u^2-1)}, \end{aligned}$$

where

$$J_2 := 7u^2(\ell - qu^2)z^2 + 4s(11u^2 - 4) + 7qu^2(1 - u^2) - 24Hs(1 - u^2)z.$$

Multiplying by $-4qHus^2(u^2-1)$, the four terms $r_j w^j$ become

$$\begin{aligned} &4\delta(1+z^2)w^3, & qJ_2w^2, \\ &-2q^2u^2(1-u^2)(1-3H^2)w - 8Hq^2u^2(1-u^2)zw, \\ &-q^2su^2(1-u^2), \end{aligned}$$

respectively. The two terms containing z to the first power, namely $-24Hqs(1-u^2)zw^2$ from qJ_2w^2 and $-8Hq^2u^2(1-u^2)zw$, add up to

$-8Hq(1-u^2)w(3sw+qu^2)z$. Comparing with (5.10) term by term, the sum is exactly Π , so

$$(5.12) \quad X\mathcal{G} - Y\mathcal{B} = -\frac{\Pi}{4qHus^2(u^2-1)}.$$

On the graph, the left side is $w_{uH} - w_{Hu} = 0$ and the denominator is nonzero. Hence $\Pi = 0$; differentiating this identity along X gives $X\Pi = 0$. $\square$

5.2. Analyticity and algebraicity. We next pass from smooth solutions to analytic germs.

Lemma 5.4 (Analyticity of the graph). *Every smooth graph solving $z_H = (1+z^2)w/s$, $z_u = (1+z^2)\mathcal{A}$, $w_H = \mathcal{B}$, $w_u = \mathcal{G}$ is locally real analytic on $V = \{Huw(1-u^2) \neq 0\}$.*

Proof. The rational vector fields X, Y are real analytic on V , since $q \neq 0$ and $s = 1 + H^2 > 0$.

Fix $p_0 = (H_0, u_0, z_0, w_0)$ on the graph. The local flows Φ_X^t, Φ_Y^r are analytic in time and initial data [23, Ch. I, §1], and, X and Y being locally Lipschitz, every C^1 integral curve coincides with the corresponding flow line. Hence

$$\Psi(h, v) := \Phi_Y^{v-u_0}(\Phi_X^{h-H_0}(p_0))$$

is analytic near (H_0, u_0) .

Since $(XH, Xu) = (1, 0)$ and $(YH, Yu) = (0, 1)$, the (H, u) -projection of $\Psi(h, v)$ is (h, v) .

The graph curve at $u = u_0$ solves the X -equation through p_0 . Uniqueness gives

$$\Phi_X^{h-H_0}(p_0) = (h, u_0, z(h, u_0), w(h, u_0)), \quad h \text{ near } H_0.$$

For fixed h , the curve $v \mapsto (h, v, z(h, v), w(h, v))$ solves the Y -equation with initial point $\Phi_X^{h-H_0}(p_0)$. Uniqueness again gives $\Psi(h, v) = (h, v, z(h, v), w(h, v))$.

Thus z, w are analytic. Uniqueness was applied only to integral curves lying on the graph, so the relation $[X, Y] = 0$ is not used. $\square$

Fix a real point (H_0, u_0) of the coordinate domain of Ω_* . By Lemma 5.4, z and w extend holomorphically to a complex polydisc Δ around (H_0, u_0) on which $H, u, w, 1-u^2$ and s have no zeros; each identity in (5.3)–(5.8), holding on the real square $\Delta \cap \mathbb{R}^2$, holds on Δ by the identity theorem in each variable. Inside the field of meromorphic germs at (H_0, u_0) set $E = \mathbb{C}(H, u)(w, z)$; $\mathbb{C}(H, u)$ embeds in it, since no nonzero polynomial vanishes on an open set. All subsequent identities hold in this field.

We now eliminate z by a resultant, to obtain a nontrivial relation $\mathcal{R}(H, u, w) = 0$.

Put $D_0 := Hqu^2s(u^2-1)$, $M := D_0\mathcal{B}$ and $\mathcal{N} := D_0X\Pi$, and write $\Pi_0 = \Pi|_{u=0}$, $\mathcal{N}_0 = \mathcal{N}|_{u=0}$.

Lemma 5.5 (Values at $u = 0$). *The functions M and $\mathcal{N}$ belong to $\mathbb{C}[H, u, w, z]$, with*

$$(5.13) \quad \mathcal{N} = D_0 \Pi_H + Hqu^2(u^2 - 1)(1 + z^2)w \Pi_z + M \Pi_w,$$

and

$$(5.14) \quad \Pi_0 = 8sw^2 \Lambda(z), \quad \Lambda(z) = 3\ell w(1 + z^2) - 3Hqz - 2q, \quad \mathcal{N}_0 = -4sw^3 \partial_w \Pi_0.$$

Proof. Substituting $C = (4s + q(1 - u^2))/(4s(1 - u^2))$ in (5.9) and multiplying by D_0 gives the polynomial

$$\begin{aligned} M &= 4s(u^2 - 1)w^3 + qu^2(u^2 - 1)[w^2(3Hz + 1) + w(1 - H^2)] \\ &\quad + 2u^2[4s + q(1 - u^2)]w^3(1 + z^2). \end{aligned}$$

Since $X = \partial_H + (1 + z^2)\frac{w}{s}\partial_z + \mathcal{B}\partial_w$ and $D_0(1 + z^2)w/s = Hqu^2(u^2 - 1)(1 + z^2)w$, the formula for $\mathcal{N}$ follows, and it is a polynomial. As $\deg_z \Pi = 2$ and $\deg_z M = 2$, it has $\deg_z \mathcal{N} \leq \max(2, 2 + 1, 2 + 2) = 4$. At $u = 0$: $\delta(0) = 6s\ell$, so the five groups in (5.10) become $24s\ell w^3(1 + z^2)$, $-16qsw^2$, $-24Hqsw^2z$, 0 and 0, whose sum is $8sw^2\Lambda(z)$. In the formula for $\mathcal{N}$ the coefficients D_0 and $Hqu^2(u^2 - 1)(1 + z^2)w$ vanish at $u = 0$, while $M|_{u=0} = -4sw^3$; since setting $u = 0$ commutes with ∂_w , this gives $\mathcal{N}_0 = -4sw^3\partial_w \Pi_0$. $\square$

By (5.10) and the proof of Lemma 5.5, $\deg_z \Pi = 2$ and $\deg_z \mathcal{N} \leq 4$. Let $\mathcal{R} := \text{Res}_z^{(2,4)}(\Pi, \mathcal{N})$ be the Sylvester determinant of Π and $\mathcal{N}$ formed with the nominal degrees $(2, 4)$ in z .

Lemma 5.6 (Nonvanishing of the resultant). *The resultant $\mathcal{R}$ is a nonzero element of $\mathbb{C}[H, u, w]$, and $\mathcal{R} = c_1 \Pi + c_2 \mathcal{N}$ for some $c_1, c_2 \in \mathbb{C}[H, u, w, z]$.*

Proof. We work in $\mathbb{C}[H, u, w, z]$, with $q \in \mathbb{C}^*$ fixed. The Sylvester determinant of two polynomials of nominal degrees $(2, 4)$ in z lies in the ideal they generate, whether or not the leading coefficients vanish [24, Ch. IV, §8], [10, Ch. 3]; this gives c_1, c_2 . Setting $u = 0$ commutes with the determinant, so $\mathcal{R}|_{u=0} = \text{Res}_z^{(2,4)}(\Pi_0, \mathcal{N}_0)$, with $\Pi_0, \mathcal{N}_0$ given by (5.14).

Write $\Lambda(z) = 3\ell w(z - \mu_1)(z - \mu_2)$ over $\mathbb{C}(H, w)$; then Π_0 has exact degree 2 in z , with leading coefficient $a_2 = 24s\ell w^3$, and for nominal degrees $(2, 4)$ the root formula $\text{Res}_z^{(2,4)}(\Pi_0, \mathcal{N}_0) = a_2^4 \mathcal{N}_0(\mu_1)\mathcal{N}_0(\mu_2)$ holds whatever the actual degree of $\mathcal{N}_0$ [24, Ch. IV, §8]. Since $\partial_w \Pi_0 = 16sw\Lambda + 24s\ell w^2(1 + z^2)$, we get $\mathcal{N}_0(\mu_j) = -96s^2\ell w^5(1 + \mu_j^2)$, and $(1 + \mu_1^2)(1 + \mu_2^2) = \Lambda(i)\Lambda(-i)/(3\ell w)^2 = q^2(4 + 9H^2)/(9\ell^2 w^2)$, because $\Lambda(\pm i) = -q(2 \pm 3iH)$. Hence

$$\mathcal{R}|_{u=0} = 2^{22} 3^4 q^2 s^8 \ell^4 (4 + 9H^2) w^{20} \neq 0,$$

as $q \neq 0$ and $s, \ell = 4 + 4H^2 + q, 4 + 9H^2$ are nonzero polynomials in the indeterminate H . $\square$

Lemma 5.7 (Algebraicity). *For a real solution with $w \neq 0$, w is algebraic over $\mathbb{C}(H, u)$ and z is algebraic over $\mathbb{C}(H, u, w)$. Hence $[E : \mathbb{C}(H, u)] < \infty$.*

Proof. By Lemma 5.6, $\mathcal{R} = c_1\Pi + c_2\mathcal{N}$ with polynomial c_1, c_2 . On the graph $\Pi = 0$ and $\mathcal{N} = D_0 X\Pi = 0$, so $\mathcal{R}(H, u, w(H, u)) = 0$ on an open polydisc. Consider $\deg_w \mathcal{R}$. *Case* $\deg_w \mathcal{R} = 0$. Then $\mathcal{R}$ is a nonzero element of $\mathbb{C}[H, u]$ vanishing on an open polydisc in the independent coordinates (H, u) , which is impossible. *Case* $\deg_w \mathcal{R} \geq 1$. Then $\mathcal{R}(H, u, Z)$ is a nonzero polynomial in Z over $\mathbb{C}(H, u)$, and $\mathcal{R}(H, u, w) = 0$ shows that w is algebraic over $\mathbb{C}(H, u)$.

For z , collect powers of z in Π : $\Pi = A_2 z^2 + A_1 z + A_0$ with

$$\begin{aligned} A_2 &= 4\delta w^3 + 7qu^2(\ell - qu^2)w^2, \\ A_1 &= -8Hq(1 - u^2)w(3sw + qu^2), \\ A_0 &= 4\delta w^3 + q[4s(11u^2 - 4) + 7qu^2(1 - u^2)]w^2 \\ &\quad - 2q^2u^2(1 - u^2)(1 - 3H^2)w - q^2su^2(1 - u^2). \end{aligned}$$

If $A_1 \neq 0$ we are done. If $A_1 = 0$, then $Hq(1 - u^2)w \neq 0$ gives $w = -qu^2/(3s)$, an element of F ; then A_0 is an element of F as well, and we show that it is not zero by expanding it in powers of u in $\mathbb{C}(H)((u))$. Since $\delta(0) = 6s\ell$, the four terms of A_0 have, with $w = -qu^2/(3s)$, lowest powers u^6, u^4, u^4 and u^2 respectively; the last term is $-q^2su^2(1 - u^2)$, with coefficient $-q^2s \neq 0$ of u^2 . Hence $A_0 = -q^2su^2 + O(u^4) \neq 0$. Thus Π is a nonzero polynomial in z over $\mathbb{C}(H, u, w)$. If its coefficients of positive degree in z both vanished, then $\Pi = A_0 \neq 0$, contradicting $\Pi = 0$; hence Π has positive degree in z , and $\Pi = 0$ shows that z is algebraic over $\mathbb{C}(H, u, w)$. Finiteness of $[E : \mathbb{C}(H, u)]$ follows, since E is generated by the two algebraic elements w, z . $\square$

5.3. Exclusion of the algebraic branches. Put $F = \mathbb{C}(H, u)$ and let $\mathbb{K} = \overline{\mathbb{C}(H)}$ be an algebraic closure of $\mathbb{C}(H)$. The derivation d/dH of $\mathbb{C}(H)$ extends uniquely to $\mathbb{K}$ (Fact A.2(a)); we denote it by a prime. Its field of constants is $\mathbb{C}$ (Fact A.2(b)):

$$(5.15) \quad \ker(d/dH|_{\mathbb{K}}) = \mathbb{C}.$$

Let

$$(5.16) \quad \mathcal{L} := \bigcup_{n \geq 1} \mathbb{K}((u^{1/n}))$$

be the field of Puiseux series $\sum_{r \geq r_0} a_r u^r$, $a_r \in \mathbb{K}$, $r \in \frac{1}{n}\mathbb{Z}$. Since $\mathbb{K}$ is algebraically closed of characteristic zero, $\mathcal{L}$ is algebraically closed by the Newton–Puiseux theorem [25], [28, Ch. IV]. This is the reason for passing from $\mathbb{C}(H)$ to $\mathbb{K}$: with coefficients in $\mathbb{C}(H)$ the corresponding field is not algebraically closed (for instance, $Z^2 = H$ has no root in $\bigcup_n \mathbb{C}(H)((u^{1/n}))$), and the embedding of Lemma 5.8 might not exist. For $0 \neq \Phi \in \mathcal{L}$ let $\nu(\Phi)$ be the smallest exponent occurring in Φ , and $\nu(0) = +\infty$; then $\nu(\Phi\Psi) = \nu(\Phi) + \nu(\Psi)$ and $\nu(\Phi + \Psi) \geq \min\{\nu(\Phi), \nu(\Psi)\}$, with equality if $\nu(\Phi) \neq \nu(\Psi)$.

Define on $\mathcal{L}$

$$(5.17) \quad D_H\left(\sum_r a_r u^r\right) := \sum_r a'_r u^r, \quad D_u\left(\sum_r a_r u^r\right) := \sum_r r a_r u^{r-1}.$$

Both are derivations (the product rule holds coefficientwise). Under the Laurent expansion at $u = 0$, $F \hookrightarrow \mathbb{C}(H)((u)) \subset \mathcal{L}$, they restrict to ∂_H, ∂_u on F : if $j : F \hookrightarrow \mathcal{L}$ denotes this embedding, then $D_H \circ j$ and $j \circ \partial_H$ (and likewise $D_u \circ j$ and $j \circ \partial_u$) are derivations $F \rightarrow \mathcal{L}$ that vanish on $\mathbb{C}$ and agree on H and u , hence agree on $\mathbb{C}[H, u]$ and, by the quotient rule, on F . Directly from (5.17): if $\nu(\Phi) = r$, write $\Phi = au^r(1 + \eta)$ with $a \in \mathbb{K}^\times$ and $\nu(\eta) > 0$; then $D_u \eta$ has order > -1 , and

$$(5.18) \quad \nu(D_H \Phi) \geq r, \quad \frac{D_u \Phi}{\Phi} = \frac{r}{u} + \frac{D_u \eta}{1 + \eta}, \quad \nu\left(\frac{D_u \eta}{1 + \eta}\right) > -1.$$

In particular, $\nu(D_u \Phi) = r - 1$ if $r \neq 0$.

Since F has characteristic zero, every algebraic extension of F is separable, and by Fact A.2(a) the derivations ∂_H, ∂_u of F extend uniquely to any algebraic extension. The next lemma states that an F -embedding intertwines these unique extensions.

Lemma 5.8 (Transport of the derivations). *Let $E \supset F$ be a finite extension on which ∂_H, ∂_u act as derivations extending those of F , and let $\iota : E \hookrightarrow \mathcal{L}$ be an F -embedding (it exists since $\mathcal{L}$ is algebraically closed). Then*

$$D_H(\iota a) = \iota(\partial_H a), \quad D_u(\iota a) = \iota(\partial_u a) \quad (a \in E).$$

Proof. Let $\mathcal{M} \in F[Z]$ be the monic minimal polynomial of a ; it is separable (characteristic zero), so $\mathcal{M}_Z(a) \neq 0$ and, ι being injective and F -linear, $\mathcal{M}_Z(\iota a) = \iota(\mathcal{M}_Z(a)) \neq 0$. Let ∂ be ∂_H or ∂_u , with counterpart D on $\mathcal{L}$, and let $\mathcal{M}^\partial$ be $\mathcal{M}$ with ∂ applied to its coefficients. Applying ∂ to $\mathcal{M}(a) = 0$ in E , and D to $\mathcal{M}(\iota a) = 0$ in $\mathcal{L}$ (where $D = \partial$ on F), gives

$$\partial a = -\frac{\mathcal{M}^\partial(a)}{\mathcal{M}_Z(a)}, \quad D(\iota a) = -\frac{\mathcal{M}^\partial(\iota a)}{\mathcal{M}_Z(\iota a)} = \iota\left(-\frac{\mathcal{M}^\partial(a)}{\mathcal{M}_Z(a)}\right) = \iota(\partial a). \quad \square$$

Apply this to $E = \mathbb{C}(H, u)(w, z)$, which is finite over F by Lemma 5.7 and stable under ∂_H, ∂_u by (5.3)–(5.8), whose denominators are products of powers of $H, u, w, s, 1 - u^2$, all invertible in E (germs not vanishing identically on the connected Δ are invertible). The derivations of E are the unique extensions of those of F (Fact A.2(a)). Identifying E with $\iota(E)$, (5.3)–(5.8) and $\Pi = X\Pi = 0$ hold in $\mathcal{L}$ with D_H, D_u , and $w \neq 0, 1 + z^2 \neq 0$ persist.

Remark 5.9 (The passage to $u = 0$ is formal). From now on u is an indeterminate of $F = \mathbb{C}(H, u)$, and “ $u = 0$ ” refers only to the valuation ν of $\mathcal{L}$. No point of the surface with $\langle N, \xi \rangle = 0$ is used, the surface is not extended, and no convergence of Puiseux series is claimed. Since H, u are independent indeterminates, H is transcendental over $\mathbb{C}$; this makes the final identities (5.21) and (5.24) contradictory.

Lemma 5.10 (Incompatibility of the orders). *Let $q \in \mathbb{C} \setminus \{0\}$. There are no $w, z \in \mathcal{L}$ with $w \neq 0, 1 + z^2 \neq 0$, satisfying (5.3)–(5.5) (with D_H, D_u) and $\Pi = 0$.*

Proof. All orders refer to the valuation ν of $\mathcal{L}$ (Remark 5.9). Write $w = tu^\gamma + \dots$ with $t \in \mathbb{K}^*$, $\gamma = \nu(w) \in \mathbb{Q}$. The cases are summarized in Table 3; in each, the leading terms are compared exactly, including possible cancellations.

Case	Orders forced	Contradiction	Equation used; nonzero leading coefficient
$\nu(z) \geq 0$, $\gamma = 0$	—	term t/u in (5.5) unmatched	(5.5); $t \neq 0$ ($t \in \mathbb{K}^*$)
$\nu(z) \geq 0$, $\gamma \geq 2$	—	$\nu(\Pi) = 2$, so $\Pi \neq 0$	Π ; coefficient $-q^2s$ of u^2 in the last term of (5.10) ($q \neq 0$, $s = 1 + H^2 \neq 0$)
$\nu(z) \geq 0$, $0 < \gamma < 2$	$\gamma = 1$, $z = c + z_1u + \dots$, $c \in \mathbb{C}$, $c \neq \pm i$	order 2 of Π and order 1 of (5.3) give (5.21)	(5.5) ($\gamma = 1$, as $t \neq 0$), Π , (5.3); $z_1 \neq 0$, $\varpi \neq 0$ by (5.19); constant term 3 in (5.21)
$\nu(z) = -b < 0$, $\gamma > 2$	—	term of order $1 - \gamma < -1$ in (5.22) unmatched	(5.22); coefficient $q/(4st) \neq 0$ of $u^{1-\gamma}$
$\nu(z) = -b < 0$, $\gamma < 2$	$\gamma - b = 1$, $\gamma + b = 2$, so $(\gamma, b) = (\frac{3}{2}, \frac{1}{2})$	$\nu(\Pi) = 2$, so $\Pi \neq 0$	(5.22), (5.4), Π ; coefficient $-q^2s$ of u^2 in the last term of (5.10)
$\nu(z) = -b < 0$, $\gamma = 2$	$b = 2$, cancellation of the leading terms of $\mathcal{A}$	orders -1 of (5.22) and -2 of (5.3) give (5.24)	(5.4): $D_u z$ has leading coefficient $-2c \neq 0$, forcing the cancellation; H^4 -coefficient 4 in (5.24)

TABLE 3. The cases in the proof of Lemma 5.10.

Preliminary: $\gamma \geq 0$, and $\gamma \geq b$ if $\nu(z) = -b < 0$. By (5.3), $w = s D_H z / (1 + z^2)$. If $\nu(z) = -b < 0$, then $\nu(D_H z) \geq -b$ and $\nu(1 + z^2) = -2b$, so $\gamma \geq b$. If $\nu(z) \geq 0$, write $z = c + \varepsilon$ with $c \in \mathbb{K}$, $\nu(\varepsilon) > 0$. When $1 + c^2 \neq 0$, $\nu(1 + z^2) = 0$ and $\gamma \geq \nu(D_H z) \geq 0$. When $c = \pm i$ (a constant), $D_H z = D_H \varepsilon$ and $1 + z^2 = \varepsilon(2c + \varepsilon)$ has order exactly $\nu(\varepsilon)$, while $\nu(D_H \varepsilon) \geq \nu(\varepsilon)$; again $\gamma \geq 0$.

Case 1: $\nu(z) \geq 0$, $z = c + \dots$ with $c \in \mathbb{K}$.

Step 1: $\gamma = 1$. If $\gamma = 0$, write $w = t + \eta$, $\nu(\eta) > 0$. In (5.5), $D_u w = D_u \eta$ has order > -1 , whereas w/u has the term t/u and the other two terms of

$\mathcal{G}$ have order ≥ 1 : a contradiction. If $\gamma \geq 2$, the first four terms of Π (those containing w) have orders at least 3γ , 2γ , $\min\{2\gamma, \gamma + 2\}$, $\gamma + 2$, all > 2 , while $-q^2su^2(1-u^2)$ has order exactly 2; so $\Pi \neq 0$. Hence $0 < \gamma < 2$. Then the three terms of $\mathcal{G}$ have orders $\gamma - 1$, ≥ 1 , $\geq \gamma + 1$, and comparing the terms of order $\gamma - 1$ in (5.5) gives $\gamma t = t$, that is, $\gamma = 1$.

Step 2: expansions. The right side of (5.3) has order $\geq \gamma = 1$, so the constant term of $D_H z$ vanishes: $c' = 0$, and $c \in \mathbb{C}$ by (5.15). Below, $o(1)$ denotes terms of positive order. Since $w/u = t + o(1)$,

$$\mathcal{A} = -\frac{qH}{4s(w/u)} + uzC = -\frac{qH}{4st} + o(1).$$

If $c = \pm i$, then $\varepsilon := z - c \neq 0$ has $\nu(D_u \varepsilon) = \nu(\varepsilon) - 1$, whereas $(1 + z^2)\mathcal{A} = \varepsilon(2c + \varepsilon)\mathcal{A}$ has order $\nu(\varepsilon)$, contradicting (5.4). So $1 + c^2 \neq 0$, and (5.4) gives $D_u z = z_1 + o(1)$ with

$$z_1 = -\frac{(1 + c^2)qH}{4st} \neq 0.$$

As $D_u(z - c)$ has no terms of negative order, $z - c$ has no terms of order in $(0, 1)$, and its coefficient of u is z_1 : $z = c + z_1 u + (\text{terms of order } > 1)$.

Step 3: coefficient identities. Insert $w = tu + (\text{order } > 1)$ and the expansion of z into Π . The terms of order 2 come from only three terms: $-16qst^2$ from the second group (the bracket equals $-16s + O(u^2)$), $-24Hqst^2c$ from the third (only $3sw^2z$ matters), and $-q^2s$ from the last. Hence

$$(5.19) \quad -qs[8t^2(2 + 3Hc) + q] = 0.$$

Comparing the coefficients of u in (5.3) gives $z'_1 = (1 + c^2)t/s$; with the value of z_1 and $c' = 0$,

$$(5.20) \quad \left(-\frac{qH}{4st}\right)' = \frac{t}{s}.$$

Set $\varpi = 2 + 3Hc$. By (5.19), $t^2 = -q/(8\varpi)$ and $\varpi \neq 0$; differentiating $t^2\varpi = -q/8$ gives $t'/t = -3c/(2\varpi)$, hence $(st)'/(st) = 2H/s - 3c/(2\varpi)$ and

$$\left(-\frac{qH}{4st}\right)' = -\frac{q}{4st} \left[1 - \frac{2H^2}{s} + \frac{3Hc}{2\varpi}\right].$$

Multiplying (5.20) by $-4st/q$ and using $-4t^2/q = 1/(2\varpi)$,

$$1 - \frac{2H^2}{s} + \frac{3Hc}{2\varpi} = \frac{1}{2\varpi}.$$

Multiplying by $2\varpi s$ and expanding,

$$(5.21) \quad 3 - 5H^2 + (9H - 3H^3)c = 0.$$

Since $c \in \mathbb{C}$ and H is transcendental over $\mathbb{C}$, this is a polynomial identity in H with constant term $3 \neq 0$, a contradiction. This excludes Case 1.

Case 2: $\nu(z) = -b < 0$, $z = cu^{-b} + \dots$, $c \in \mathbb{K}^*$. By the preliminary, $\gamma \geq b > 0$. Divide (5.5) by w and multiply (5.4) by $z/(1 + z^2)$; adding, the

terms $\pm qHuz/(4sw)$ cancel and $-uC(1+z^2) + uz^2C = -uC$, so

$$(5.22) \quad \frac{D_u w}{w} + \frac{zD_u z}{1+z^2} = \frac{1}{u} - uC + \frac{qu}{4sw}.$$

The left side is $\frac{1}{2}D_u\Phi/\Phi$ with $\Phi = w^2(1+z^2)$, $\nu(\Phi) = 2\gamma - 2b$; by (5.18) it equals $(\gamma - b)/u + (\text{order} > -1)$. On the right, $-uC$ has order exactly 1 (as $C(0) = \ell/(4s) \neq 0$), and $qu/(4sw)$ has order $1 - \gamma$. If $\gamma > 2$, the latter is a term of order < -1 that nothing on the left can match. So $\gamma \leq 2$.

For $\gamma > b$, the first term of $\mathcal{A}$ has order $1 - \gamma$ and the second $1 - b$, so $\nu(\mathcal{A}) = 1 - \gamma$, and comparing orders in (5.4) ($\nu(D_u z) = -b - 1$) gives

$$(5.23) \quad -b - 1 = 1 - \gamma - 2b, \quad \text{i.e.} \quad \gamma + b = 2.$$

Subcase $\gamma < 2$. Then $qu/(4sw)$ has order > -1 , and the coefficients of u^{-1} in (5.22) give $\gamma - b = 1$. So $\gamma > b$, and (5.23) yields $b = \frac{1}{2}$, $\gamma = \frac{3}{2}$. The five terms of Π in (5.10) then have orders $\frac{7}{2}$, 3 , $\geq \frac{5}{2}$, $\frac{7}{2}$ and exactly 2 (the bracket in the second term has constant term $-16s$), so $\nu(\Pi) = 2$ and $\Pi \neq 0$, a contradiction.

Subcase $\gamma = 2$. If $b < 2$, then $\gamma > b$, so (5.23) gives $b = 0$, which is excluded; since $b \leq \gamma = 2$, $b = 2$. Write $w = tu^2 + \dots$, $z = cu^{-2} + \dots$. Both terms of $\mathcal{A}$ have order -1 , so $(1+z^2)\mathcal{A}$ has order -5 unless their leading coefficients cancel, whereas $\nu(D_u z) = -3$. Cancellation requires

$$-\frac{qH}{4st} + \frac{\ell c}{4s} = 0, \quad \text{i.e.} \quad c = \frac{qH}{\ell t}.$$

In (5.22) the coefficient of u^{-1} is $\gamma - b = 0$ on the left and $1 + q/(4st)$ on the right, so $t = -q/(4s)$ and $c = -4sH/\ell$. The coefficient of u^{-2} in (5.3) gives $c' = c^2 t/s = -4qH^2/\ell^2$. On the other hand, with $s' = 2H$ and $\ell' = 8H$,

$$c' = -\frac{4[(1+3H^2)\ell - 8H^2s]}{\ell^2}.$$

Equating, $(1+3H^2)\ell - 8H^2s = qH^2$; expanding with $\ell = 4 + 4H^2 + q$ this reads

$$(5.24) \quad 4s^2 + q(1+2H^2) = 0.$$

Its H^4 -coefficient is $4 \neq 0$, impossible because H is transcendental over $\mathbb{C}$. All cases are excluded. □

5.4. Conclusion of the proof.

Proof of Theorem 1.1 for $k \neq 0$. If Ω were nonempty, Lemma 3.5 would give a point of Ω_* with coordinates (H, u) , near which z, w satisfy (5.3)–(5.8) and $\Pi = X\Pi = 0$, with $w \neq 0$ by (5.6) and $1+z^2 = \sec^2 \alpha \geq 1$. Lemmas 5.4, 5.7 and 5.8 transport these identities to $\mathcal{L}$, contradicting Lemma 5.10. Hence $dH = 0$; connectedness makes H constant. The converse follows from (3.9). □

6. THE PRODUCT SPACES

In this section $\tau = 0$, $k = \kappa$, and $\xi = \partial_t$ is parallel. We write Q_{AR} for the same formal expression (1.1) with $\tau = 0$, that is, $Q_{AR} = 2Hp - \kappa A^2$; the product spaces lie outside the scope of Theorem 1.1. Data (λ, u, H, p, A) on a simply connected domain satisfying (3.6)–(3.7) define a local immersion into $M^2(\kappa) \times \mathbb{R}$: for $\kappa \neq 0$ these are the integrability conditions (C.1)–(C.4) of [20, Theorem 2.3, p. 283], which are sufficient when $\kappa - 4\tau^2 \neq 0$ (see also [11, Theorem 4.3, p. 101]); for $\kappa = 0$ one adds the Gauss equation and the vertical equation $A_\zeta - (\lambda_\zeta/\lambda)A = up$ (the pairing of $\nabla_{\partial_\zeta} T$ with ∂_ζ in (3.3), using (3.8)) and uses the classical fundamental theorem for surfaces in $\mathbb{R}^3$ [14, §4-3 and the Appendix to Ch. 4]; the vertical equations make $T + uN$ a parallel unit field, identified with ∂_t by a rigid motion. Surfaces with constant angle function in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ were classified in [12] and [13]; we single out those with holomorphic Q_{AR} . Among surfaces of constant angle $u_0 \notin \{0, \pm 1\}$, holomorphy of Q_{AR} is equivalent to ∇H being parallel to T on $\{dH \neq 0\}$, that is, to H being constant along the lines of curvature orthogonal to T (Corollary 6.4). This is the condition, additional to the data of the classifications in [12, 13], that selects the family of Proposition 6.1: there all data depend only on the parameter σ along T .

Fix $\kappa \in \mathbb{R}$, $u_0 \in (-1, 1) \setminus \{0\}$, and put $R_0 = \sqrt{1 - u_0^2}$. Let $Y(\sigma) > 0$ be a solution of $(\log Y)'' = -\kappa u_0^2 Y^2$ on an interval; this equation has the first integral

$$(6.1) \quad \left(\frac{Y'}{Y}\right)^2 = C - \kappa u_0^2 Y^2, \quad C \in \mathbb{R} \text{ constant.}$$

On the strip $\zeta = \sigma + iv$ put $v = \log Y$ and

$$(6.2) \quad \lambda = Y^2, \quad u = u_0, \quad A = \frac{R_0 Y}{2}, \quad H = \frac{R_0 v'}{2u_0 Y}, \quad p = -\frac{\lambda H}{2}.$$

Proposition 6.1 (Constant angle surfaces). *The data (6.2) define a local immersion into $M^2(\kappa) \times \mathbb{R}$ with constant angle function u_0 ,*

$$(6.3) \quad Q_{AR} \equiv -\frac{R_0^2 C}{4u_0^2}, \quad H' = -\frac{R_0 C}{2u_0 Y}, \quad 4H^2 + \kappa R_0^2 = \frac{R_0^2 C}{u_0^2 Y^2}.$$

In particular, Q_{AR} is holomorphic, and the immersion is CMC if and only if $C = 0$. Moreover:

- (i) *if $\kappa > 0$, then $C > 0$;*
- (ii) *if $\kappa = 0$, then $C \geq 0$, and $C = 0$ exactly when Y is constant and $H = p = 0$ (a plane of constant angle);*
- (iii) *if $\kappa < 0$, every sign of C occurs; the surfaces with $C < 0$ satisfy $4H^2 < |\kappa| R_0^2$.*

Proof. By hypothesis $v'' = -\kappa u_0^2 Y^2$, and (6.1) holds with a constant C because $(v'^2 + \kappa u_0^2 e^{2v})' = 2v'(v'' + \kappa u_0^2 e^{2v}) = 0$. All data depend on σ only, so $F_\zeta = F_{\bar{\zeta}} = F'/2$, with $' = d/d\sigma$, and we check (3.6)–(3.7), the Gauss equation and the vertical equation.

- $A_{\bar{\zeta}} = \frac{u\lambda}{2}H$, i.e. $A' = u_0\lambda H$: both sides equal $R_0Yv'/2$.
- $p_{\bar{\zeta}} = \frac{\lambda}{2}(H_{\zeta} + \kappa uA)$, i.e. $p' = \lambda(H'/2 + \kappa u_0A)$: $H' = \frac{R_0}{2u_0Y}(v'' - v'^2) = -\frac{R_0C}{2u_0Y}$, and $p' = -\frac{R_0}{4u_0}(Yv')' = -\frac{R_0Y}{4u_0}(v'^2 + v'') = -\frac{R_0YC}{4u_0} + \frac{\kappa u_0R_0Y^3}{2} = \lambda(\frac{H'}{2} + \kappa u_0A)$.
- $u_{\zeta} = -HA - \frac{2p}{\lambda}\bar{A}$: both sides vanish, since $2p/\lambda = -H$ and A is real.
- $4|A|^2 = R_0^2Y^2 = \lambda(1 - u_0^2)$.
- Gauss: $\det S = H^2 - 4|p|^2/\lambda^2 = 0$, and $(\log \lambda)'' = 2v'' = -2\kappa u_0^2\lambda$, which is Liouville's formula for $K = \kappa u_0^2$.
- Vertical equation: $A_{\zeta} - (\lambda_{\zeta}/\lambda)A = \frac{R_0Yv'}{4} - \frac{R_0Yv'}{2} = -\frac{R_0Yv'}{4} = u_0p$.

Finally $Q_{AR} = 2Hp - \kappa A^2 = -\lambda H^2 - \kappa R_0^2Y^2/4 = -\frac{R_0^2}{4u_0^2}(v'^2 + \kappa u_0^2Y^2) = -\frac{R_0^2C}{4u_0^2}$, and the last identity in (6.3) follows from $4H^2 = R_0^2v'^2/(u_0^2Y^2)$ and (6.1). If $\kappa > 0$, (6.1) gives $C \geq \kappa u_0^2Y^2 > 0$. If $\kappa = 0$, then $C = v'^2 \geq 0$, and $C = 0$ if and only if Y is constant, in which case $H = p = 0$. If $\kappa < 0$ and $C > 0$, examples are given by Proposition 6.2 below; for $C < 0$, by homothety we may take $\kappa = -1$, and $Y = \sqrt{|C|}/(|u_0| \cos(\sqrt{|C|}\sigma))$ on $|\sqrt{|C|}\sigma| < \pi/2$ satisfies $Y'/Y = \sqrt{|C|} \tan(\sqrt{|C|}\sigma)$ and $(Y'/Y)^2 = |C|/\cos^2(\sqrt{|C|}\sigma) - |C| = u_0^2Y^2 + C$; finally $C < 0$ in (6.3) gives $4H^2 < |\kappa|R_0^2$. $\square$

Fix $\mathfrak{a} \neq 0$, put $\mathfrak{m} = \sqrt{1 + \mathfrak{a}^2}$, and let $(y(\sigma), r(\sigma))$, with $y > 0$, solve

$$(6.4) \quad y' = -\mathfrak{a}y r, \quad r' = \mathfrak{a}\kappa y^2, \quad r^2 + \kappa y^2 = 1.$$

The constraint is preserved, since $(r^2 + \kappa y^2)' = 2\mathfrak{a}\kappa y^2 r - 2\mathfrak{a}\kappa y^2 r = 0$. For $\kappa \leq 0$, $r^2 \geq 1$, so r has a fixed sign, which may be either; for $\kappa > 0$, r may change sign. On the strip $\zeta = \sigma + iv$ consider the data

$$(6.5) \quad u = \frac{\mathfrak{a}}{\mathfrak{m}}, \quad \lambda = \mathfrak{m}^2 y^2, \quad H = -\frac{r}{2\mathfrak{m}y}, \quad p = -\frac{\lambda H}{2} = \frac{\mathfrak{m}y r}{4}, \quad A = \frac{y}{2}.$$

Proposition 6.2 (The hypothesis $\tau \neq 0$ is essential). *For every $\kappa \in \mathbb{R}$, the data (6.5) define a local immersion into $M^2(\kappa) \times \mathbb{R}$ with $Q_{AR} \equiv -1/4$ and $H' = -\mathfrak{a}/(2\mathfrak{m}y) \neq 0$. In particular, Q_{AR} is holomorphic and the mean curvature is not constant.*

Proof. These are the surfaces of Proposition 6.1 with $u_0 = \mathfrak{a}/\mathfrak{m}$, $R_0 = 1/\mathfrak{m}$, $Y = \mathfrak{m}y$ and $C = \mathfrak{a}^2$. Indeed $v' = y'/y = -\mathfrak{a}r$, so $(Y'/Y)^2 = \mathfrak{a}^2(1 - \kappa y^2) = C - \kappa u_0^2 Y^2$; then $A = R_0 Y/2 = y/2$, $H = R_0 v'/(2u_0 Y) = -r/(2\mathfrak{m}y)$, $Q_{AR} = -R_0^2 C/(4u_0^2) = -1/4$ and $H' = -R_0 C/(2u_0 Y) = -\mathfrak{a}/(2\mathfrak{m}y) \neq 0$. $\square$

Explicit solutions of (6.4), with $x := \mathfrak{a}\sigma + \sigma_0$, are: $y = y_0 e^{-\mathfrak{a}\sigma}$, $r = 1$ ($y_0 > 0$, all σ) for $\kappa = 0$; $y = \kappa^{-1/2} \operatorname{sech} x$, $r = \tanh x$, for all $x \in \mathbb{R}$, when $\kappa > 0$; and $y = |\kappa|^{-1/2} \operatorname{csch} x$, $r = \coth x$, on $x > 0$, when $\kappa < 0$. For $\kappa > 0$ the mean curvature $H = -\sqrt{\kappa} \sinh x/(2\mathfrak{m})$ vanishes exactly on the curve $x = 0$, where $r = 0$, and $H' \neq 0$ there. For $\kappa = \pm 1$, restricted to $x > 0$,

this is, up to notation, the family of [20, Example 3.1, p. 286], where the original differential $P dz^2 = -\frac{1}{4} dz^2$ is observed to be holomorphic although H is not constant; for other $\kappa \neq 0$ it follows by a homothety. The examples are therefore known; what is new is that constant angle is necessary and that the family of Proposition 6.1 exhausts all possibilities (Theorem 6.3). They have zero extrinsic curvature, as required by [8, p. 173]. Their normalized differential $Q_{\text{FM}} = Q_{\text{AR}}/H = -1/(4H)$ is defined on $\{H \neq 0\}$ and is not holomorphic there; when $\kappa > 0$ it does not extend continuously across the curve $H = 0$, since $Q_{\text{AR}} = -1/4 \neq 0$, while Q_{AR} is holomorphic on the whole surface. Thus, at $\tau = 0$, the original and normalized differentials have different exceptional surfaces (for Q_{FM} in $H^2 \times \mathbb{R}$, the rotational surfaces of [20, Example 3.2]); the nonvanishing of τ is used in Lemma 3.5 and in the round proof. By (6.3), the surfaces of Proposition 6.2 are, after a change of conformal parameter, exactly those of Proposition 6.1 with $C > 0$ and the same u_0 (end of Step 5 below); for $\kappa < 0$, those with $C < 0$ satisfy $4H^2 < |\kappa|R_0^2$ and are not among them.

In the following theorem, ∇H , u and T are as in §3, now with $\tau = 0$ and $k = \kappa$.

Theorem 6.3 (Classification in the product spaces). *Let $\kappa \neq 0$, and let Σ be a surface in $M^2(\kappa) \times \mathbb{R}$ with holomorphic Q_{AR} and nonconstant mean curvature. Then dH vanishes nowhere on Σ , and:*

- (a) *the angle function is a constant $u_0 \notin \{0, \pm 1\}$;*
- (b) *∇H is a principal direction with principal curvature 0, and T is parallel to ∇H ;*
- (c) $|\nabla H| = \frac{|u_0| |4H^2 + \kappa(1 - u_0^2)|}{2\sqrt{1 - u_0^2}};$
- (d) *Σ is locally congruent to a surface of Proposition 6.1 with the same u_0 and with $C \neq 0$ of the sign of $4H^2 + \kappa(1 - u_0^2)$; up to local congruence it is determined by κ , u_0 and this sign. When $\kappa > 0$ or $4H^2 > |\kappa|(1 - u_0^2)$, it is, after a change of conformal parameter, locally congruent to a surface of Proposition 6.2 with the same u_0 , including near points where $H = 0$ if $\kappa > 0$.*

Proof. Outline. Steps 1–3 show that $du = 0$ on a dense open subset of $\Omega = \{dH \neq 0\}$, Step 4 deduces (a)–(c), Step 5 identifies each component of Ω with a model, and Step 6, the only global step, shows $\Omega = \Sigma$ by Grönwall's inequality.

Reduction. Holomorphy is again equivalent to (3.10) with $\tau = 0$, and the proof of Proposition 3.3 applies verbatim: on Ω , in the frame $e_1 = \nabla H/f$, $e_2 = Je_1$, $f = |\nabla H|$,

$$[S]_{(e_1, e_2)} = \begin{pmatrix} 0 & 0 \\ 0 & 2H \end{pmatrix}, \quad \det S = 0, \quad K = \kappa u^2.$$

This proves the first half of (b). The proof of Lemma 3.4 uses $\tau \neq 0$ only through $\rho^2 = H^2 + \tau^2 \neq 0$; on the open set $\Omega_0 := \Omega \cap \{H \neq 0, 0 < |u| < 1\}$

it therefore gives, with $\tau = 0$, $P = H \sin \theta$, $L = H \cos \theta$, $R = \sqrt{1 - u^2}$ and $\nabla_{e_1} e_1 = a e_2$, $\nabla_{e_2} e_1 = b e_2$ (here a, b are the connection coefficients of Lemma 3.4),

$$(6.6) \quad \begin{aligned} a &= -\frac{\kappa u R \sin \theta}{2H}, & b &= -\frac{f}{H} - \frac{\kappa u R \cos \theta}{2H}, \\ e_1 u &= 0, & e_2 u &= -2HR \sin \theta, \\ e_1 \theta &= -a, & e_2 \theta &= -b + \frac{2Hu \cos \theta}{R}, & e_2 f &= af. \end{aligned}$$

These formulas only require $H \neq 0$ and $|u| < 1$; in particular they hold where $u = 0$.

Step 1: the branch $\sin \theta \equiv 0$. Let V be an open set with $H \neq 0$, $|u| < 1$ and $\sin \theta \equiv 0$. Then $\cos \theta = \varepsilon \in \{\pm 1\}$ is locally constant, $T = \varepsilon R e_1$, $a = 0$, and $e_1 u = e_2 u = 0$, so u is locally constant on V . The equation for $e_2 \theta$ in (6.6) gives $b = 2\varepsilon H u / R$, and comparing with the expression of b ,

$$(6.7) \quad f = -\frac{\varepsilon u (4H^2 + \kappa R^2)}{2R}.$$

In particular $u \neq 0$ on V , since $f > 0$. The relation $e_2 f = af = 0$ holds because f is a function of H , and the Gauss equation is automatic: $K = -e_1 b - b^2 = u^2(4H^2 + \kappa R^2)/R^2 - 4H^2 u^2/R^2 = \kappa u^2$.

Step 2: Ω_0 is dense in Ω . The sets $\{H = 0\}$, $\{|u| = 1\}$ and $\{u = 0\}$ are closed in Ω ; we show they have empty interior. If $H \equiv 0$ on an open set, then $dH = 0$ there. If $|u| \equiv 1$ on an open set, then $T \equiv 0$ and (3.3) gives $uS \equiv 0$, so $S \equiv 0$ and $H \equiv 0$. If $u \equiv 0$ on an open set, shrink it so that $H \neq 0$; there $R = 1$ and $0 = e_2 u = -2H \sin \theta$, so $\sin \theta \equiv 0$, and Step 1 gives $u \neq 0$, a contradiction.

Step 3: $\sin \theta = 0$ on Ω_0 . Suppose $\sin \theta \neq 0$ on a nonempty open set $U \subset \Omega_0$. There $dH \wedge du(e_1, e_2) = -2fHR \sin \theta \neq 0$, so (H, u) are local coordinates, with $e_1 = f \partial_H$ and $e_2 = -2HR \sin \theta \partial_u$. Put $c = \cos \theta$ and $\mathbf{n} = 4H^2 + \kappa R^2$. Dividing (6.6) by $e_1 H = f$ and $e_2 u = -2HR \sin \theta$,

$$(6.8) \quad f_u = \frac{\kappa u}{4H^2} f, \quad c_H = -\frac{\kappa u R (1 - c^2)}{2H f}, \quad c_u = \frac{2fR + u c \mathbf{n}}{4H^2 R^2}.$$

(For the first, $a/(-2HR \sin \theta) = \kappa u/(4H^2)$; for the last two, $c_H = -\sin \theta \theta_H$ and $c_u = -\sin \theta \theta_u$.)

The Gauss equation gives f_H . With $\omega = a e_1^b + b e_2^b$, $d\omega = -K dA$ reads $K = e_2 a - e_1 b - a^2 - b^2$. Using (6.8),

$$\begin{aligned} e_2 a &= \kappa(1 - 2u^2)(1 - c^2) - \frac{\kappa u c (2fR + u c \mathbf{n})}{4H^2}, \\ a^2 &= \frac{\kappa^2 u^2 R^2 (1 - c^2)}{4H^2}, & b^2 &= \frac{(2f + \kappa u R c)^2}{4H^2}, \\ e_1 b &= -\frac{f f_H}{H} + \frac{f^2}{H^2} + \frac{\kappa^2 u^2 R^2 (1 - c^2)}{4H^2} + \frac{\kappa u R c f}{2H^2}. \end{aligned}$$

Substituting in $K = \kappa u^2$ and collecting terms, where the terms in $c^2 \mathbf{n}$ give $\kappa u^2 c^2 + \kappa^2 u^2 R^2 c^2 / (4H^2)$, we obtain

$$(6.9) \quad f_H = \frac{\mathcal{E}}{Hf}, \quad \mathcal{E} := 2f^2 + 2\kappa u R c f + \frac{\kappa^2 u^2 R^2}{2} + \kappa H^2 (3u^2 - 1 + R^2 c^2).$$

Compatibility of f . From (6.8), $f_{uH} = \kappa u f_H / (4H^2) - \kappa u f / (2H^3)$, while (6.9) gives $f_{Hu} = \mathcal{E}_u / (Hf) - \kappa u \mathcal{E} / (4H^3 f)$. Equating and multiplying by $4H^3 f$,

$$(6.10) \quad 2H^2 \mathcal{E}_u - \kappa u \mathcal{E} + \kappa u f^2 = 0.$$

Differentiating $\mathcal{E}$ with (6.8), $(uR)_u = (1 - 2u^2)/R$ and $(R^2)_u = -2u$, the terms in c^2 cancel and one finds

$$\mathcal{E}_u = \frac{2\kappa u f^2}{H^2} + 3\kappa R c f + \frac{\kappa^2 u^2 R c f}{H^2} + \kappa^2 u (1 - 2u^2) + 6\kappa H^2 u + \frac{\kappa^2 u R^2 c^2}{2}.$$

Substituting in (6.10), the terms in c^2 and in $u^2 R c f$ cancel, and after division by $\kappa/2 \neq 0$,

$$(6.11) \quad 12H^2 R f c = -u\mathcal{U}, \quad \mathcal{U} := 6f^2 + 24H^4 + 2\kappa H^2 (3 - 7u^2) - \kappa^2 u^2 R^2.$$

Elimination of c . Since $12H^2 R f \neq 0$, (6.11) gives c . Differentiating it in u , with f_u from (6.8),

$$\left(\frac{u}{Rf} \right)_u = \frac{4H^2 - \kappa u^2 R^2}{4H^2 R^3 f}, \quad \mathcal{U}_u = \frac{3\kappa u f^2}{H^2} - 28\kappa H^2 u - 2\kappa^2 u (1 - 2u^2),$$

and comparing with the value of c_u in (6.8), in which

$$u c \mathbf{n} = -\frac{u^2 \mathcal{U} \mathbf{n}}{12H^2 R f},$$

we obtain, after multiplication by $48H^4 R^3 f$ and division by R^2 ,

$$\mathcal{U} (2\kappa u^2 - 4H^2) - 4H^2 u \mathcal{U}_u - 24H^2 f^2 = 0.$$

Expanding, the terms $\pm 12\kappa u^2 f^2$ cancel, and

$$(6.12) \quad \begin{aligned} 24H^2 f^2 = \Phi := & -48H^6 - 12\kappa H^4 + 108\kappa H^4 u^2 + 12\kappa^2 H^2 u^2 \\ & - 24\kappa^2 H^2 u^4 + \kappa^3 u^4 (u^2 - 1). \end{aligned}$$

Contradiction. By (6.8), $(f^2)_u = \kappa u f^2 / (2H^2)$, so (6.12) gives $2H^2 \Phi_u = \kappa u \Phi$ on U . For each fixed H this is an identity between polynomials in u on an interval; the right side has degree 7 in u , with leading coefficient κ^4 , while the left side has degree at most 5. As $\kappa \neq 0$, this is impossible. Hence $\sin \theta \equiv 0$ on Ω_0 .

Step 4: conclusion of (a)–(c). By Steps 1 and 3, $du = 0$ on the dense open set Ω_0 , hence on Ω , and $u \equiv u_0$ on each component of Ω ; by Step 2, $u_0 \notin \{0, \pm 1\}$. This is (a); $T = \varepsilon R_0 e_1$ gives the second half of (b), and (6.7) with $f > 0$ gives (c) on Ω_0 , hence on Ω by continuity.

Step 5: congruence. Fix a component Ω_1 of Ω , with constant angle u_0 . By (c), $\mathbf{n}_0 := 4H^2 + \kappa R_0^2$ does not vanish on Ω_1 , so it has a constant sign $\varsigma \in \{\pm 1\}$, and (6.7) gives $\varepsilon = -\varsigma \text{sign } u_0$. On Ω_1 , $e_2 H = 0$ and $[e_1, e_2] =$

$-be_2$, where $b = 2\varepsilon H u_0 / R_0$ and f are functions of H alone. Hence, near any point, there is a coordinate t with $e_1 = f(H)\partial_H$ and $e_2 = \mu(H)\partial_t$, $\mu'/\mu = -b/f$, and the metric $dH^2/f^2 + dt^2/\mu^2$, the shape operator ($Se_1 = 0$, $Se_2 = 2He_2$), $T = \varepsilon R_0 e_1$ and $u = u_0$ are given, as functions of (H, t) , by formulas depending only on u_0 , ς and κ (the constant in μ is absorbed into t).

Now take a surface of Proposition 6.1 with the same u_0 and with sign $C = \varsigma$; by Proposition 6.1 and (6.3) this is possible exactly when $\varsigma = 1$ or $\kappa < 0$, which covers all cases allowed by (c). It has holomorphic $\mathcal{Q}_{\text{AR}}$, constant angle u_0 , and $H' = -R_0 C / (2u_0 Y) \neq 0$, so H is strictly monotone in σ , and Steps 1–4 apply to it with the same u_0 and ς , hence the same ε . Its mean curvature fills the whole interval of values allowed by ς . Indeed, on a maximal solution of $v'' = -\kappa u_0^2 e^{2v}$: if $\kappa > 0$ (so $C > 0$), Y increases to its maximum, where $v' = 0$ and $H = 0$, and then decreases, with $Y \rightarrow 0$ and $v' \rightarrow \mp\sqrt{C}$ at the ends, so H runs over all of $\mathbb{R}$; if $\kappa < 0$ and $C > 0$, v' never vanishes, Y runs over $(0, \infty)$, and $4H^2 = R_0^2(C/(u_0^2 Y^2) + |\kappa|)$ runs over $(|\kappa|R_0^2, \infty)$, with the sign of H fixed by that of v'/u_0 , which can be chosen; if $\kappa < 0$ and $C < 0$, Y has a minimum where $H = 0$ and tends to ∞ at both ends, so $4H^2$ runs over $[0, |\kappa|R_0^2)$ and H over $(-h_0, h_0)$, $h_0 = \sqrt{|\kappa|}R_0/2$. Hence the model surface contains points with any value of H taken on Ω_1 , and, in the coordinates (H, t) above, both surfaces have the same metric, shape operator, T and u . By the uniqueness part of the fundamental theorem [11, Theorem 4.3, p. 101], Ω_1 is locally congruent to the model. When $\kappa > 0$, or $4H^2 > |\kappa|R_0^2$, we have $C > 0$; the model is then a surface of Proposition 6.2 with the same u_0 . Indeed, put $\mathbf{a} := u_0/R_0$, so that $\mathbf{m} = 1/R_0$ and $u_0 = \mathbf{a}/\mathbf{m}$, and change the conformal parameter by $\zeta = b\tilde{\zeta}$, $b := |\mathbf{a}|/\sqrt{C}$; this is a change of coordinates on the surface, not a homothety of the ambient space. Then $\tilde{Y}(\tilde{\sigma}) := bY(b\tilde{\sigma})$ satisfies $(\log \tilde{Y})'' = -\kappa u_0^2 \tilde{Y}^2$ with constant $\tilde{C} = b^2 C = \mathbf{a}^2$ in (6.1). Setting $y := \tilde{Y}/\mathbf{m}$ and $r := -\tilde{v}'/\mathbf{a}$, with $\tilde{v} = \log \tilde{Y}$, one checks $r^2 = 1 - \kappa y^2$, $y' = -\mathbf{a}yr$ and $r' = \kappa u_0^2 \tilde{Y}^2/\mathbf{a} = \mathbf{a}\kappa y^2$, so (y, r) solves (6.4), and the data (6.2) become (6.5), with $H = -r/(2\mathbf{m}y)$. No change of orientation is needed, since r is allowed to take either sign. This includes the points where $H = 0$, which occur exactly when $\kappa > 0$.

Steps 1–5 establish (a)–(d) on each connected component of Ω .

Step 6: $\Omega = \Sigma$. Since H is not constant, $\Omega \neq \emptyset$. Suppose $\Omega \neq \Sigma$, and let U be a component of Ω , with constant angle u_0 and $R_0 = \sqrt{1 - u_0^2}$. By (c),

$$|\nabla H| = c_0 |n_0(H)| \quad \text{on } U, \quad c_0 = \frac{2|u_0|}{R_0}, \quad n_0(H) = H^2 + \frac{\kappa R_0^2}{4}.$$

Since Σ is connected, there is a piecewise smooth path $\gamma : [0, 1] \rightarrow \Sigma$, with $|\gamma'| \leq m$ for a constant $m > 0$, such that $\gamma(0) \in U$ and $\gamma(1) \notin U$. Let $s_* = \inf\{s : \gamma(s) \notin U\}$; as U is open, $\gamma([0, s_*)) \subset U$ and $x := \gamma(s_*) \notin U$. Then $dH(x) = 0$: otherwise $x \in \Omega$, and a connected neighborhood of x in Ω meets U , so $x \in U$. Letting $s \rightarrow s_*$, the continuity of H and $|\nabla H|$ gives $n_0(H(x)) = 0$. If $\kappa > 0$ this is impossible. If $\kappa < 0$, then $H(x) = \epsilon_1 h_0$ with

$h_0 = \sqrt{|\kappa|}R_0/2$ and $\epsilon_1 = \pm 1$. Put $\chi(s) = H(\gamma(s)) - \epsilon_1 h_0$. The function χ is Lipschitz, hence absolutely continuous, and differentiable except at the finitely many break points of γ . Near s_* we have $|H(\gamma(s)) + \epsilon_1 h_0| \leq 3h_0$, hence

$$\begin{aligned} |\chi'(s)| &\leq m |\nabla H|(\gamma(s)) = mc_0 |\chi(s)| |H(\gamma(s)) + \epsilon_1 h_0| \\ &\leq M_0 |\chi(s)|, \quad M_0 := 3mc_0 h_0, \end{aligned}$$

on $[s_* - \eta_0, s_*]$ for some $\eta_0 > 0$, except at finitely many points. As $\chi(s_*) = 0$, absolute continuity gives $|\chi(s)| \leq \int_{s_*}^s M_0 |\chi(r)| dr$, and Grönwall's inequality gives $\chi \equiv 0$ on $[s_* - \eta_0, s_*]$. Thus $H = \epsilon_1 h_0$ at the points $\gamma(s) \in U$, $s \in [s_* - \eta_0, s_*]$, where (c) gives $|\nabla H| = 0$, contradicting $U \subset \Omega$. Hence $\Omega = \Sigma$, that is, dH vanishes nowhere, and $u \equiv u_0$ on Σ . No completeness is used. $\square$

Theorem 1.2 is Theorem 6.3: item (d) of the former follows from Step 5, where the metric, the shape operator, T and u are expressed in terms of κ , u_0 and the sign of $4H^2 + \kappa R_0^2$ only.

Proof of Corollary 1.3. If Σ is closed, H attains its maximum at a point where $dH = 0$. By Theorem 6.3, H is constant. This agrees with [8, pp. 172–173]. $\square$

Corollary 6.4. *Let $\kappa \neq 0$ and let Σ be a surface in $M^2(\kappa) \times \mathbb{R}$ with constant angle function $u \equiv u_0 \notin \{0, \pm 1\}$. Then $\mathcal{Q}_{\text{AR}}$ is holomorphic if and only if ∇H is parallel to T on $\{dH \neq 0\}$, that is, H is constant along the lines of curvature orthogonal to T .*

Proof. Since u is constant, (3.4) with $\tau = 0$ gives $\langle SX, T \rangle = -X(u) = 0$ for all X , so $ST = 0$; as $T \neq 0$, T is a principal direction with principal curvature 0, and the other principal direction JT has principal curvature $2H$. On $\{dH \neq 0\}$, the argument of Proposition 3.7 with $\tau = 0$ (where $D = \overline{\nabla}$) shows that holomorphy is equivalent, pointwise, to $S\nabla H = 0$. Where $H \neq 0$, the kernel of S is spanned by T , so $S\nabla H = 0$ means $\nabla H \parallel T$; since $\{H \neq 0\}$ is dense in $\{dH \neq 0\}$ and both conditions are closed, holomorphy on $\{dH \neq 0\}$ is equivalent to $\nabla H \parallel T$ there. On the interior of $\{dH = 0\}$, H is locally constant and $(Q_{\text{AR}})_{\bar{\zeta}} = 0$ by (3.9); by continuity $(Q_{\text{AR}})_{\bar{\zeta}}$ then vanishes everywhere. The converse is part (b) of Theorem 6.3. $\square$

Remark 6.5. In the surfaces of Proposition 6.1 with $\kappa < 0$, H approaches $\pm h_0$ only where $Y \rightarrow \infty$, that is, at infinite distance, in accordance with Step 6 of the proof of Theorem 6.3. For $\kappa = 0$, $\mathcal{Q}_{\text{AR}} = 2Hp$, and holomorphy of Hp for surfaces in $\mathbb{R}^3$ is a classical problem outside the scope of this paper.

7. PROOF OF COROLLARY 1.4

By Theorem 1.1, H is constant; compactness is used only to count zeros. With isothermal parameters [9], Σ is a compact Riemann surface of genus $\mathbf{g}$, and since p and A transform as coefficients of a $(2, 0)$ - and a $(1, 0)$ -form, $\mathcal{Q}_{\text{AR}}$ is a holomorphic section of $K_{\Sigma}^{\otimes 2}$, of degree $4\mathbf{g} - 4$ [19, Ch. III]. If $\mathcal{Q}_{\text{AR}} \not\equiv 0$

it has $4\mathbf{g} - 4$ zeros counted with multiplicity; for $\mathbf{g} = 0$ this is impossible, so $\mathcal{Q}_{\text{AR}} \equiv 0$, and the rotational invariance follows from the references given after Corollary 1.4. For $k = 0$ it is direct: $p \equiv 0$, so Σ is a totally umbilical, hence round, sphere of $S^3(\tau^2)$. $\square$

APPENDIX A. AUXILIARY COMPUTATIONS

This appendix contains two facts used in Section 5: Fact A.1 is the direct (but lengthy) computation of the coefficients in the proof of Lemma 5.3, and Fact A.2 collects standard facts on derivations of algebraic extensions.

Fact A.1 (Coefficients of the compatibility identity). *With $a_{-1}, a_0, g_1, g_0, b_3, b_2, b_1$ as in the proof of Lemma 5.3, the coefficients $r_0, \dots, r_3$ of (5.11) are*

$$\begin{aligned} r_0 &= -\frac{qu}{4Hs}, & r_1 &= -\frac{qu(1 - 3H^2 + 4Hz)}{2Hs^2}, \\ r_2 &= -\frac{J_2}{4Hus^2(u^2 - 1)}, & r_3 &= -\frac{\delta(1 + z^2)}{qHus^2(u^2 - 1)}, \end{aligned}$$

with J_2 as defined after (5.11) in the main text.

Proof. Put $v := (1 + z^2)/s$, $V_0 := 1 + z^2$ and $d := 1 - u^2$. Since $(b_1)_z = 0$, $r_0 = (g_0)_H - g_0b_1$. The quotient rule gives

$$\begin{aligned} (g_0)_H &= \frac{qu}{4} \cdot \frac{zs - (Hz + 1) \cdot 2H}{s^2} = \frac{qu}{4s^2} [z(s - 2H^2) - 2H] \\ &= \frac{qu}{4s^2} [z(1 - H^2) - 2H]. \end{aligned}$$

Since $g_0b_1 = qu(Hz + 1)(1 - H^2)/(4Hs^2)$,

$$\begin{aligned} r_0 &= \frac{qu}{4Hs^2} [H(z(1 - H^2) - 2H) - (Hz + 1)(1 - H^2)] \\ &= \frac{qu}{4Hs^2} [Hz(1 - H^2) - 2H^2 - Hz(1 - H^2) - (1 - H^2)] \\ &= \frac{qu}{4Hs^2} (-s) = -\frac{qu}{4Hs}, \end{aligned}$$

where $2H^2 + (1 - H^2) = s$. For the other coefficients, $C_H = -qH/(2s^2)$, $C_u = 2u/d^2$, and $(b_1)_u = (b_1)_z = (b_2)_u = 0$. First,

$$\begin{aligned} r_1 &= (g_1)_H + v(g_0)_z - V_0a_{-1}(b_2)_z - 2g_0b_2 \\ &= \frac{quHV_0}{2s^2} + \frac{quHV_0}{4s^2} + \frac{3quHV_0}{4s^2} - \frac{qu(Hz + 1)(3Hz + 1)}{2Hs^2} \\ &= \frac{qu}{2Hs^2} [3H^2(1 + z^2) - (Hz + 1)(3Hz + 1)] \\ &= -\frac{qu(1 - 3H^2 + 4Hz)}{2Hs^2}. \end{aligned}$$

Next, using $(b_3)_z = -16Cz/(qH)$ and $(b_2)_z = 3/s$, the terms containing zCV_0/s are $v(g_1)_z = -2uzCV_0/s$, $-V_0a_0(b_2)_z = -3uzCV_0/s$ and

$-V_0 a_{-1}(b_3)_z = -4uzCV_0/s$, whose sum is $-9uzCV_0/s$; the remaining terms are $-g_1 b_2$ and $-3g_0 b_3$, expanded with $g_1 = 1/u - uCV_0$. Hence

$$\begin{aligned} r_2 &= v(g_1)_z - V_0[a_{-1}(b_3)_z + a_0(b_2)_z] - g_1 b_2 - 3g_0 b_3 \\ &= -\frac{9uzCV_0}{s} - \frac{3Hz+1}{Hsu} + \frac{uCV_0(3Hz+1)}{Hs} \\ &\quad - \frac{3(Hz+1)}{Hsu} + \frac{6uCV_0(Hz+1)}{Hs} \\ &= \frac{7uCV_0 - (6Hz+4)/u}{Hs} \\ &= \frac{7u^2(4s+qd)(1+z^2) - 4sd(6Hz+4)}{4Hus^2d}. \end{aligned}$$

Here $4s+qd = \ell - qu^2$, and the constant term in the numerator is $7u^2(4s+qd) - 16sd = 4s(11u^2-4) + 7qu^2d$. Finally, $(b_3)_u = -8/(qHu^3) - 8C_uV_0/(qH)$ gives

$$\begin{aligned} r_3 &= -(b_3)_u - V_0 a_0(b_3)_z - 2g_1 b_3 \\ &= \frac{1}{qH} \left[\frac{8}{u^3} + 8C_uV_0 + 16uz^2C^2V_0 \right. \\ &\quad \left. - 2\left(\frac{1}{u} - uCV_0\right)\left(\frac{4}{u^2} - 8CV_0\right) \right] \\ &= \frac{1}{qH} \left[8C_uV_0 + \frac{24CV_0}{u} + 16uC^2V_0(z^2 - V_0) \right] \\ &= \frac{8V_0}{qH} \left(C_u + \frac{3C}{u} - 2uC^2 \right), \end{aligned}$$

since the terms $\pm 8/(qHu^3)$ cancel, $-2 \cdot \frac{1}{u} \cdot (-8CV_0) - 2(-uCV_0)\frac{4}{u^2} = \frac{24CV_0}{u}$, and $z^2 - V_0 = -1$. Expanding the last bracket,

$$\begin{aligned} C_u + \frac{3C}{u} - 2uC^2 &= \frac{2u}{d^2} + \frac{3}{ud} + \frac{3q}{4su} - \frac{2u}{d^2} - \frac{qu}{sd} - \frac{q^2u}{8s^2} \\ &= \frac{24s^2 + 6qs - 14qsu^2 - q^2u^2 + q^2u^4}{8s^2ud} = \frac{\delta}{8s^2ud}. \end{aligned}$$

Collecting the four coefficients gives the stated formulas for $r_0, \dots, r_3$. $\square$

Fact A.2. *Let F be a field of characteristic zero with a derivation D .*

- (a) *If a is algebraic over F with minimal polynomial $\mathcal{M}$, then $\mathcal{M}_Z(a) \neq 0$ and D extends uniquely to a derivation of $F(a)$, with*

$$D(a) = -\frac{\mathcal{M}^D(a)}{\mathcal{M}_Z(a)},$$

where $\mathcal{M}^D$ applies D to the coefficients of $\mathcal{M}$. Consequently D extends uniquely to any algebraic extension of F .

- (b) *For $F = \mathbb{C}(H)$ and $D = d/dH$, the field of constants of the extension to $\mathbb{K} = \overline{\mathbb{C}(H)}$ is $\mathbb{C}$.*

Proof. (a) This is standard [24, Ch. VIII, §5]. Applying D to $\mathcal{M}(a) = 0$ gives $\mathcal{M}^D(a) + \mathcal{M}_Z(a)D(a) = 0$, which forces the formula. Conversely, $\mathcal{M}_Z$ is invertible modulo $\mathcal{M}$ ($\mathcal{M}$ is irreducible and $\deg \mathcal{M}_Z < \deg \mathcal{M}$), so the formula defines a derivation of $F[Z]$ into $F[Z]/(\mathcal{M})$ that kills $\mathcal{M}$, hence descends to $F(a) \cong F[Z]/(\mathcal{M})$. Uniqueness at each finite stage gives compatibility along towers, hence the extension to algebraic extensions.

(b) If $a/b \in \mathbb{C}(H)$ is reduced and $(a/b)' = 0$, then $a'b = ab'$ and $a \mid a'$; as $\deg a' < \deg a$ unless a is constant, a is constant, and then $ab' = 0$ makes b constant. So the constants of $\mathbb{C}(H)$ are $\mathbb{C}$. Now let $c \in \mathbb{K}$ with $c' = 0$, with minimal polynomial $\mathcal{M} = Z^m + \sum_{j < m} a_j Z^j$ over $\mathbb{C}(H)$. By (a), $0 = \mathcal{M}^D(c) = \sum_{j < m} a'_j c^j$, a polynomial relation of degree $< m$; by minimality all $a'_j = 0$, so $a_j \in \mathbb{C}$. An irreducible polynomial over $\mathbb{C}(H)$ with coefficients in $\mathbb{C}$ has degree 1 ($\mathbb{C}$ being algebraically closed), so $m = 1$ and $c \in \mathbb{C}$. $\square$

DATA AVAILABILITY STATEMENT

This manuscript has no associated data.

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interest.

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