

ASYMPTOTICALLY CONICAL SHRINKING ENDS OF THE r -MEAN CURVATURE FLOW: ROTATIONAL CONSTRUCTION AND UNIQUENESS

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ABSTRACT. For $1 \leq r \leq n-1$ we study shrinking self-similar ends of the r -mean curvature flow, $S_r = c\langle X, N \rangle$ with $c < 0$, that are asymptotic to a cone. For every rotational cone of positive slope we construct a smooth embedded rotational end and prove that its profile is the only positive solution of the profile equation with the prescribed slope at infinity. Its angular defect $xu' - u$ is strictly negative, which yields r -admissibility, and the profile has a complete asymptotic expansion for $u(x)/x$ in powers of $x^{-(r+1)}$, valid after arbitrary differentiation. The profiles depend real analytically on the slope and, on sufficiently distant exterior regions, foliate the image of the corresponding family. Our main result removes the symmetry assumption: two ends that converge smoothly, with multiplicity one, to the same circular cone coincide outside a compact set. The proof transforms the equation for their difference into an almost-eigenfunction inequality for a drift Laplacian on an auxiliary conical metric and applies Bernstein's asymptotic analysis on weakly conical ends. The argument extends to strictly r -admissible cones whose links have constant principal curvatures.

1. INTRODUCTION

Let $X_0 : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ be an immersed hypersurface with principal curvatures $\lambda_1, \dots, \lambda_n$. For $1 \leq r \leq n$, let

$$S_r(\lambda) = \sum_{i_1 < \dots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}$$

be the r th elementary symmetric function. The r -mean curvature flow is the curvature evolution

$$(1.1) \quad \left(\frac{\partial X}{\partial t} \right)^\perp = S_r N.$$

For $r = 1$ this is the mean curvature flow. For $r > 1$ the flow is fully nonlinear and is parabolic on the Gårding cone

$$\Gamma_r = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_r(\lambda) > 0\}.$$

The study of contracting convex hypersurfaces by homogeneous curvature functions goes back to Andrews [3–5]; see also [6, 7, 11, 23, 24] for powers of curvature functions, pinching, convergence, and self-similar solutions. The restriction to an ellipticity cone is intrinsic to this theory: outside it the linearization of the speed need not be positive.

If $X(\cdot, t) = \phi(t)X_0$ is a homothetic solution of (1.1), then X_0 satisfies

$$(1.2) \quad S_r = c\langle X, N \rangle$$

for a constant c . Conversely, an r -admissible solution of (1.2) generates the homothetic solution with

$$\phi(t) = (1 + (r+1)ct)^{1/(r+1)}.$$

Thus $c < 0$ corresponds to a shrinker with extinction time $-1/((r+1)c)$. Under $X \mapsto \rho X$ the coefficient changes according to $c \mapsto \rho^{-(r+1)}c$.

For $r = 1$, self-shrinkers model Type-I singularities of the mean curvature flow; see Huisken [17, 18] and Colding–Minicozzi [13]. Rotational examples have played a prominent role since Angenent's shrinking torus [8]. Kleene and Møller [21, Theorem 1, p. 3943] constructed a one-parameter family of properly embedded rotational self-shrinking ends

2020 *Mathematics Subject Classification.* Primary 53E10; Secondary 53C21, 53C42, 34B40.

Key words and phrases. r -mean curvature flow, self-shrinker, asymptotically conical end, uniqueness at infinity, rotational hypersurface, Gårding cone.

asymptotic to rotational cones, usually called trumpets, and classified the noncompact rotational self-shrinking ends [21, Theorem 3, p. 3947]. Wang [27, Theorem 1.3, p. 614] proved that a smooth properly embedded self-shrinking end asymptotic to a regular cone is unique, without any symmetry assumption, and Bernstein [10] later gave an elliptic proof of Wang's theorem, based on the asymptotic structure of almost eigenfunctions of drift Laplacians on weakly conical ends. Khan [20, Theorem 1.1, p. 56] extended this uniqueness to properly immersed self-shrinkers of higher codimension, also via Bernstein's method. Conical self-shrinkers also enter the gluing constructions of Kapouleas–Kleene–Møller [19] and the analysis of asymptotically conical tangent flows by Chodosh–Schulze [12].

For fully nonlinear speeds, most of the noncompact theory concerns convex solutions, Gauss curvature flows, or translating solitons. For positive powers K^α of the Gauss curvature, Urbas showed that every complete noncompact convex contracting self-similar solution is a hyperplane [26, Theorem 2.1, p. 255], and that convex expanding solutions exist inside, and are asymptotic to the boundary of, every convex cone containing no line, possibly in a generalized sense [26, Theorem 1, p. 252]; rotational translators for general extrinsic flows were studied by Rengaswami [25]. Gao, Li and Ma proved that, for $1 \leq k \leq n-1$ and $\alpha \geq 1/k$, strictly convex closed self-similar solutions of the σ_k^α -flow are round spheres [14, Theorem 1.4, p. 2]. Guo constructed rotational conical shrinkers for elliptic symmetric speeds that are homogeneous of degree one [15, Theorem 1.1, p. 7354], and proved their uniqueness under additional hypotheses on the speed and on the asymptotic cone [16, Theorem 2.1, p. 5]. These results do not cover (1.2) with speed S_r when $r > 1$: S_r has degree r , and replacing it by $S_r^{1/r}$ changes both the elliptic equation and its scaling, even on the same admissibility cone. Rigidity and gap phenomena for S_r -shrinkers were studied in [1, 2, 9], and translating solitons for higher-order mean curvature flows satisfy a different soliton equation [22]. The present paper addresses the complementary problems of existence and uniqueness of asymptotically conical S_r -shrinking ends.

Uniqueness at infinity. Our main result is a uniqueness theorem at infinity, without any symmetry assumption. Let C be a regular cone over a smooth closed hypersurface $\Gamma \subset \mathbb{S}^n$, with unit normal N_C . We say that an end is *smoothly asymptotic to C with multiplicity one* if, outside a compact set, it is the normal graph

$$(1.3) \quad X_f(\rho, \theta) = \rho\theta + f(\rho, \theta)N_C(\theta), \quad |\nabla_C^j f| = o(\rho^{1-j}) \quad (j \geq 0),$$

of a function f over the part $\{\rho \geq \rho_0\}$ of C ; see Section 7. This is the hypothesis of Wang [27]: the rescalings tX converge to C in C_{loc}^∞ as $t \downarrow 0$. Let $O(n)$ act on $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ on the second factor and, for $\sigma > 0$, let

$$\mathcal{C}_\sigma = \{(x, y\omega) : x > 0, y = \sigma x, \omega \in \mathbb{S}^{n-1}\}$$

be the rotational cone of slope σ .

Theorem A. *Let $1 \leq r \leq n-1$, $c < 0$, and $\sigma > 0$. Suppose that two smooth ends satisfy (1.2) and are smoothly asymptotic, with multiplicity one, to $\mathcal{C}_\sigma$, with the orientation converging to $N_C = (\sigma, -\omega)/\sqrt{1+\sigma^2}$. Then the two ends coincide outside a compact set.*

No admissibility assumption is made on the ends in Theorem A: the principal curvatures of $\mathcal{C}_\sigma$ lie in the interior of Γ_r because $r \leq n-1$, so smooth convergence to the cone makes both ends strictly r -admissible outside a compact set.

The proof has two steps. First, the equation forces the difference of two such ends to vanish at infinity faster than every power of the distance (Lemma 7.1); flatness alone, however, does not imply equality. Second, a nonlinear radial change of variables, dictated by the degree- r homogeneity of S_r , together with the Newton tensor of the cone, produces an auxiliary conical metric on which the difference satisfies an almost-eigenfunction inequality for a drift Laplacian. The asymptotic analysis of Bernstein [10, Theorem 1.2, p. 1898] then shows that a solution of such an inequality with vanishing asymptotic trace vanishes identically outside a compact set. For $r = 1$ the construction reduces to Bernstein's proof of Wang's theorem; for $r > 1$ the effective metric and the radial drift must be normalized together. Proposition 7.6 extends the argument to strictly r -admissible cones whose links have constant principal curvatures, including noncircular examples.

The elliptic argument settles the circular case and the case of links with constant principal curvatures; uniqueness over general admissible links remains open (Question 9.5).

The rotational profile. Theorem A is a uniqueness statement. We now construct the ends to which it applies, and describe them in detail. For a positive function u of x we consider the rotational graph

$$X(x, \omega) = (x, u(x)\omega), \quad \omega \in \mathbb{S}^{n-1},$$

oriented by $N = (u', -\omega)/\sqrt{1 + (u')^2}$. Put

$$\Psi = xu' - u, \quad m = u\sqrt{1 + (u')^2}, \quad \kappa = \frac{-c}{\binom{n-1}{r-1}} > 0.$$

In Section 2 we show that (1.2) becomes the profile equation

$$(1.4) \quad u'' = \left(\frac{n-r}{ru} + \kappa m^{r-1} \Psi \right) (1 + (u')^2).$$

The quantity Ψ is the angular defect of the profile: $(u/x)' = \Psi/x^2$, and $\langle X, N \rangle = \Psi/\sqrt{1 + (u')^2}$.

Theorem B. *Let $1 \leq r \leq n-1$, $c < 0$, and $\sigma > 0$. There exists a smooth function $u_\sigma : (0, \infty) \rightarrow (0, \infty)$ solving (1.4) such that $u_\sigma(x)/x \rightarrow \sigma$ as $x \rightarrow \infty$, and it is unique in the following strong sense: every positive solution of (1.4) on a half-line (d, ∞) with $u(x)/x \rightarrow \sigma$ coincides with u_σ on the common interval of definition. Its rotational graph Σ_σ is a smooth embedded r -admissible solution of (1.2), asymptotic to $\mathcal{C}_\sigma$, and for every $x > 0$,*

$$(1.5) \quad 0 < u_\sigma(x) - \sigma x \leq \frac{n-r}{2r\kappa\sigma^r} x^{-r},$$

$$(1.6) \quad |u'_\sigma(x) - \sigma| \leq \frac{3(n-r)}{2r\kappa\sigma^r} x^{-(r+1)},$$

$$(1.7) \quad 0 > xu'_\sigma(x) - u_\sigma(x) \geq -\frac{n-r}{r\kappa\sigma^r} x^{-r}.$$

In particular, $u_\sigma(x)/x$ decreases strictly to σ and $|A_{\Sigma_\sigma}| = O(|X|^{-1})$ as $x \rightarrow \infty$. For every $b > 0$ the tail $\Sigma_\sigma \cap \{x \geq b\}$ is proper and complete in the direction $x \rightarrow \infty$.

The admissibility assertion deserves emphasis. The principal curvatures of Σ_σ are

$$\mu = -\frac{u''_\sigma}{(1 + (u'_\sigma)^2)^{3/2}}, \quad \lambda = \frac{1}{u_\sigma \sqrt{1 + (u'_\sigma)^2}},$$

where λ has multiplicity $n-1$. The r -admissibility condition is equivalent to $\mu > -\frac{n-r}{r}\lambda$, and the profile equation shows that this inequality is equivalent to $\Psi < 0$ (Proposition 2.1). Hence admissibility holds on the whole end for every slope; no smallness assumption on σ is needed. The range $r \leq n-1$ is sharp within the strictly admissible rotational graphical class (Proposition 2.2).

The profile equation (1.4) is regular at $x = 0$: its right-hand side is real analytic on $\{u > 0\}$ for every $x \in \mathbb{R}$. Thus u_σ continues to a maximal interval (x_σ, ∞) , with $x_\sigma \in [-\infty, 0]$, on which the generating curve remains a graph over the axis. Theorem B concerns the part of this interval where $x > 0$; Proposition 8.4 records what can be said about the maximal graphical interval, and Question 9.4 asks for its description.

Remark 1.1. For $r = 1$ we have $\kappa = -c$, and Theorem B produces a positive rotational graphical self-shrinking end with a prescribed conical asymptote. After matching the normalization of the shrinker coefficient, Kleene–Møller’s classification of rotational self-shrinking ends [21, Theorem 3, p. 3947] identifies it with their trumpet. Among smooth properly embedded self-shrinking ends that are asymptotic to $\mathcal{C}_\sigma$ in the sense of Wang, uniqueness outside a compact set is Wang’s theorem [27, Theorem 1.3, p. 614]; it is also recovered by Theorem A and Corollary 1.2, whose proof reduces, for $r = 1$, to Bernstein’s argument [10].

The next result describes the complete asymptotic structure of the profile. For $r = 1$ it is consistent with the ODE analysis of Kleene–Møller [21].

Theorem C. *Let u_σ be the profile in Theorem B. There is a uniquely determined sequence $(c_j)_{j \geq 1}$ such that, for all integers $N \geq 1$ and $k \geq 0$,*

$$(1.8) \quad \frac{d^k}{dx^k} \left(u_\sigma(x) - \sigma x - \sum_{j=1}^N c_j x^{1-j(r+1)} \right) = O \left(x^{1-(N+1)(r+1)-k} \right) \quad (x \rightarrow \infty).$$

The leading coefficient is

$$(1.9) \quad c_1 = \frac{n-r}{r(r+1)\kappa\sigma^r(1+\sigma^2)^{(r-1)/2}},$$

and the meridian curvature satisfies

$$(1.10) \quad \mu(x) = -\frac{n-r}{\kappa\sigma^r(1+\sigma^2)^{(r+2)/2}} x^{-r-2} + O(x^{-2r-3}).$$

Combining Theorems A, B and C we obtain a complete description of the ends asymptotic to a circular cone.

Corollary 1.2. *Let $1 \leq r \leq n-1$, $c < 0$, and $\sigma > 0$. Every smooth end satisfying (1.2) that is smoothly asymptotic, with multiplicity one, to $\mathcal{C}_\sigma$, with the orientation converging to $N_C = (\sigma, -\omega)/\sqrt{1+\sigma^2}$, coincides outside a compact set with the rotational end Σ_σ of Theorem B. In particular, it is rotationally symmetric and r -admissible outside a compact set, and it has the asymptotic expansion of Theorem C.*

Proof. By Theorem C, the end Σ_σ is smoothly asymptotic to $\mathcal{C}_\sigma$ with multiplicity one (Proposition 8.1), and its orientation converges to N_C . Theorem A applies to Σ_σ and the given end. $\square$

The asymptotic hypothesis of Theorem A is stronger than the scalar slope condition $u(x)/x \rightarrow \sigma$ that suffices for uniqueness within the rotational class (Theorem B); in exchange, no symmetry is assumed.

Dependence on the slope. The rotational profiles depend real analytically on the cone slope.

Theorem D. *For every compact interval $I \Subset (0, \infty)$ there is $R_I > 0$ such that $\sigma \mapsto u_\sigma - \sigma x$ is real analytic on a neighborhood of I with values in weighted Hölder spaces on $[R_I, \infty)$ that control $u_\sigma - \sigma x$ and all its derivatives at infinity, and $\sigma \mapsto u_\sigma \in C^{k,\alpha}([\varepsilon, E])$ is real analytic for every $k \geq 0$, $0 < \alpha < 1$, and $0 < \varepsilon < E$. Moreover, $\partial_\sigma u_\sigma(x) = x + O_I(x^{-r})$, and for slopes in the interior of I the ends $\Sigma_\sigma \cap \{x > R_I\}$ form a real-analytic foliation of their image.*

Theorem D is proved in Section 6 (Theorem 6.4 and Corollary 6.5). It does not imply a global ordering of the family: numerical computations indicate that profiles with different slopes intersect near $x = 0$ (Remark 9.1).

Figure 1 illustrates the rotational profiles and their asymptotic cones for $r = 1, 2$. This figure and Remarks 8.6 and 9.1 report numerical computations; they are illustrations, not certified bounds, and no proof in the paper uses them.

Organization. Although Theorem A is stated first, its proof in Section 7 is independent of the rotational theory, which is developed first because it is elementary and provides the model ends. In Section 2 we derive the profile equation, prove the admissibility criterion, and treat the case $r = n$. Section 3 constructs the profile by finite-radius approximation, and Section 4 proves rotational uniqueness and an explicit eventual ordering. Section 5 establishes the complete asymptotic expansion, and Section 6 proves analytic dependence on the slope and the exterior foliation. Section 7 contains the proof of Theorem A and its extension to cones whose links have constant principal curvatures. Section 8 collects geometric consequences, including the maximal graphical continuation of the profile, and Section 9 discusses open problems.

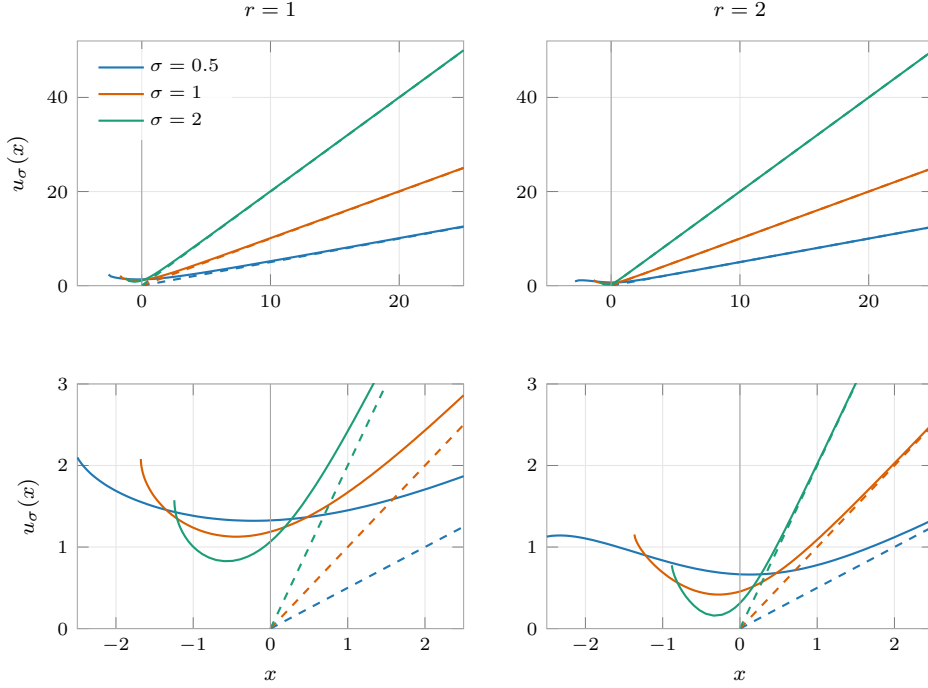


FIGURE 1. Numerical approximations of the rotational profiles u_σ for $n = 3$, $\kappa = 1$ and $\sigma = 0.5, 1, 2$, for $r = 1$ ($c = -1$, left) and $r = 2$ ($c = -2$, right). Solid curves: computed profiles, drawn until $|u'_\sigma| = 60$; dashed lines: the cones $u = \sigma x$, $x > 0$. The lower row enlarges $-2.5 \leq x \leq 2.5$. The profiles were computed from terminal data $u = \sigma x + c_1 x^{-r}$, $u' = \sigma - r c_1 x^{-r-1}$ at $x = 30$ with an implicit Runge–Kutta (Radau) method, relative tolerance 10^{-11} .

2. ROTATIONAL GEOMETRY AND THE PROFILE EQUATION

Let $O(n)$ act on $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ in the standard way on the second factor. A rotational hypersurface is parametrized by

$$X(t, \omega) = (x(t), y(t)\omega), \quad \omega \in \mathbb{S}^{n-1},$$

where $y > 0$. Put

$$W = \sqrt{(x')^2 + (y')^2}$$

and choose the unit normal

$$N = \frac{1}{W}(y', -x'\omega).$$

The meridian principal curvature and the transversal principal curvature are

$$\mu = \frac{y'x'' - y''x'}{W^3}, \quad \lambda = \frac{x'}{yW},$$

with λ of multiplicity $n - 1$. Moreover,

$$\langle X, N \rangle = \frac{xy' - yx'}{W}.$$

Thus

$$(2.1) \quad S_r = \binom{n-1}{r-1} \mu \lambda^{r-1} + \binom{n-1}{r} \lambda^r.$$

We henceforth use a graphical profile

$$x(t) = t, \quad y(t) = u(t) > 0.$$

Then $W = \sqrt{1 + (u')^2}$ and

$$(2.2) \quad \mu = -\frac{u''}{W^3}, \quad \lambda = \frac{1}{uW}, \quad \langle X, N \rangle = \frac{xu' - u}{W}.$$

As in the introduction, put

$$\Psi = xu' - u, \quad m = uW, \quad \kappa = \frac{-c}{\binom{n-1}{r-1}} > 0.$$

Using $\binom{n-1}{r} = \frac{n-r}{r} \binom{n-1}{r-1}$, equations (1.2) and (2.1) become

$$(2.3) \quad u'' = \left(\frac{n-r}{ru} + \kappa m^{r-1} \Psi \right) (1 + (u')^2).$$

This is the profile equation (1.4) of the introduction.

Proposition 2.1. *Let $u > 0$ solve (2.3). At every point at which $\Psi < 0$, the associated rotational hypersurface has a principal-curvature vector in Γ_r . Conversely, if $\lambda > 0$, r -admissibility is equivalent to $\Psi < 0$.*

Proof. For $1 \leq j \leq r$, equation (2.1) with r replaced by j gives

$$S_j = \binom{n-1}{j-1} \lambda^{j-1} \left(\mu + \frac{n-j}{j} \lambda \right).$$

Since $j \mapsto (n-j)/j$ is decreasing and $\lambda > 0$, all the inequalities $S_1 > 0, \dots, S_r > 0$ are equivalent to

$$\mu > -\frac{n-r}{r} \lambda.$$

Set

$$\Phi = \frac{n-r}{ru} + \kappa m^{r-1} \Psi.$$

By (2.3) and (2.2),

$$\mu = -\frac{\Phi}{W}, \quad \frac{n-r}{r} \lambda = \frac{n-r}{ruW}.$$

Hence,

$$\mu + \frac{n-r}{r} \lambda = -\frac{\kappa m^{r-1} \Psi}{W},$$

which is positive exactly when $\Psi < 0$. $\square$

The following proposition describes the case $r = n$ and explains the restriction in Theorem B.

Proposition 2.2. *Let $c < 0$ and suppose that a positive graphical solution of the rotational S_n -shrinker equation is defined on a right half-line and satisfies $u(x)/x \rightarrow \sigma > 0$; set $\kappa = -c > 0$. Then $u(x) = \sigma x$. Therefore, the associated cone has $S_n = 0$, and its curvature vector lies on the boundary of Γ_n ; in particular, no strictly n -admissible rotational graphical end can have a positive conical asymptote.*

Proof. For $r = n$ the first term in (2.3) vanishes and, with $W = (1 + (u')^2)^{1/2}$ and $m = uW$,

$$u'' = \kappa m^{n-1} W^2 \Psi, \quad \Psi' = xu'' = \mathcal{A}(x) \Psi, \quad \mathcal{A}(x) := \kappa x m^{n-1} W^2 > 0.$$

Hence, for any $x \geq x_0$ in the interval of definition,

$$\Psi(x) = \Psi(x_0) \exp \left(\int_{x_0}^x \mathcal{A}(s) ds \right).$$

Since $u(x)/x \rightarrow \sigma$, one has $u(x) \geq (\sigma/2)x$ for all sufficiently large x . Thus

$$\mathcal{A}(x) \geq \kappa x u(x)^{n-1} \geq \kappa (\sigma/2)^{n-1} x^n.$$

If $\Psi(x_0) \neq 0$, then Ψ has a fixed sign and grows in modulus at least like $\exp(Cx^{n+1})$. Because $(u/x)' = \Psi/x^2$, the integral of $(u/x)'$ on a right half-line would diverge, contradicting the finite limit $u(x)/x \rightarrow \sigma$. Therefore $\Psi \equiv 0$, whence $(u/x)' = 0$ and $u = \sigma x$. Its meridian curvature is zero, so $S_n = 0$ and the principal-curvature vector is not in the open Gårding cone Γ_n . $\square$

Remark 2.3. For $r = n$ the speed is the Gauss–Kronecker curvature. Proposition 2.2 is consistent with Urbas' theorem that a complete noncompact convex contracting self-similar solution of the K^α -flow is a hyperplane [26, Theorem 2.1, p. 255]. Neither statement contains the other: Proposition 2.2 assumes rotational symmetry and graphicality of an end, but neither convexity nor completeness.

3. CONSTRUCTION AND QUANTITATIVE ESTIMATES

3.1. Variation of parameters. The construction uses the following integral representation.

Lemma 3.1. *Let v solve*

$$(3.1) \quad v'' + f(x)(v - xv') = g(x)$$

on (d, a) . Then

$$(3.2) \quad \begin{aligned} v(x) = & \frac{v(a)}{a}x - (av'(a) - v(a))x \int_x^a \frac{1}{s^2} \exp\left(-\int_s^a zf(z)dz\right) ds \\ & + x \int_x^a \frac{1}{t^2} \int_t^a sg(s) \exp\left(-\int_t^s zf(z)dz\right) ds dt. \end{aligned}$$

Proof. Write $v = xy$. Then

$$xy'' + (2 - x^2f)y' = g.$$

The equation for $w = y'$ is linear. Multiplying it by the integrating factor

$$x^2 \exp\left(-\int^x zf(z)dz\right)$$

and integrating first from x to a and then once more yields (3.2). $\square$

For a fixed solution u of (2.3), equation (3.1) holds with

$$(3.3) \quad f = \kappa m^{r-1}(1 + (u')^2), \quad g = \frac{n-r}{r} \frac{1 + (u')^2}{u}.$$

3.2. Finite-radius profiles and uniform estimates. Fix $\sigma > 0$ and set $a_0 = (n-r)/r$. For every $a > 0$, let $u_{\sigma,a}$ be the solution of (2.3) with terminal data

$$u_{\sigma,a}(a) = \sigma a, \quad u'_{\sigma,a}(a) = \sigma.$$

The terminal term involving $av'(a) - v(a)$ in (3.2) vanishes. For a positive function v , set

$$\mathcal{E}_v(t, s) = \exp\left(-\kappa \int_t^s z(v(z)\sqrt{1 + (v'(z))^2})^{r-1}(1 + (v'(z))^2) dz\right).$$

Hence,

$$(3.4) \quad u_{\sigma,a}(x) = \sigma x + \frac{n-r}{r}x \int_x^a \frac{1}{t^2} \int_t^a \frac{s(1 + (u'_{\sigma,a}(s))^2)}{u_{\sigma,a}(s)} \mathcal{E}_{u_{\sigma,a}}(t, s) ds dt.$$

Lemma 3.2. *For every $a > 0$, $u_{\sigma,a}$ extends to $(0, a]$. Moreover,*

$$\Psi_{\sigma,a} < 0 \quad \text{on } (0, a)$$

and, for $0 < x \leq a$,

$$(3.5) \quad \sigma x \leq u_{\sigma,a}(x) \leq \sigma x + \frac{n-r}{2r\kappa\sigma^r}x^{-r},$$

$$(3.6) \quad |u'_{\sigma,a}(x) - \sigma| \leq \frac{3(n-r)}{2r\kappa\sigma^r}x^{-(r+1)}.$$

Proof. At $x = a$, $\Psi_{\sigma,a}(a) = 0$ and

$$\Psi'_{\sigma,a}(a) = au''_{\sigma,a}(a) > 0.$$

Thus $\Psi_{\sigma,a} < 0$ immediately to the left of a . If it had a largest zero $x_1 < a$, then $\Psi'(x_1) > 0$, forcing $\Psi > 0$ immediately to the right of x_1 , a contradiction. Therefore $\Psi_{\sigma,a} < 0$ on the maximal left interval. Hence $(u_{\sigma,a}/x)' < 0$ and $u_{\sigma,a}(x) \geq \sigma x$.

Fix x in the maximal interval. For $z \in [x, a]$,

$$m_{\sigma,a}(z) \geq u_{\sigma,a}(z) \geq \sigma z \geq \sigma x.$$

Set

$$\theta'(z) = \kappa(\sigma x)^{r-1}z(1 + (u'_{\sigma,a}(z))^2).$$

For $x \leq t \leq s \leq a$ the integrand of the inner integral in (3.4) equals

$$\frac{\theta'(s)}{\kappa(\sigma x)^{r-1}u_{\sigma,a}(s)} \mathcal{E}_{u_{\sigma,a}}(t, s) \leq \frac{\theta'(s)}{\kappa\sigma^r x^{r-1}t} \exp\left(-\int_t^s \theta'(z)dz\right),$$

because $u_{\sigma,a}(s) \geq \sigma s \geq \sigma t$ and, since $m_{\sigma,a}(z) \geq \sigma x$, the exponent of $\mathcal{E}_{u_{\sigma,a}}(t, s)$ is at most $-\int_t^s \theta'$. Since

$$\int_t^a \theta'(s) \exp\left(-\int_t^s \theta'(z) dz\right) ds \leq 1,$$

the inner integral is bounded by $1/(\kappa\sigma^r x^{r-1}t)$. Integrating t^{-3} gives (3.5). To record the derivative estimate and its constant, write

$$I_a(t) := \int_t^a \frac{s(1 + (u'_{\sigma,a}(s))^2)}{u_{\sigma,a}(s)} \mathcal{E}_{u_{\sigma,a}}(t, s) ds.$$

The preceding comparison, now made with base point t , gives

$$(3.7) \quad 0 \leq I_a(t) \leq \frac{1}{\kappa\sigma^r t^r} \quad (0 < t \leq a).$$

Formula (3.4) becomes $u_{\sigma,a}(x) - \sigma x = a_0 x \int_x^a t^{-2} I_a(t) dt$. Leibniz' rule therefore gives, including the lower-limit boundary term,

$$u'_{\sigma,a}(x) - \sigma = a_0 \int_x^a t^{-2} I_a(t) dt - a_0 \frac{I_a(x)}{x}.$$

Using (3.7) separately in the integral and boundary terms,

$$\begin{aligned} |u'_{\sigma,a}(x) - \sigma| &\leq \frac{a_0}{\kappa\sigma^r} \left(\int_x^\infty t^{-r-2} dt + x^{-r-1} \right) \\ &= \frac{a_0}{\kappa\sigma^r} \left(1 + \frac{1}{r+1} \right) x^{-r-1} \leq \frac{3a_0}{2\kappa\sigma^r} x^{-r-1}, \end{aligned}$$

because $r \geq 1$. Since $a_0 = (n-r)/r$, this is exactly (3.6).

If the maximal left endpoint were $x_{\min} > 0$, estimates (3.5)–(3.6) would keep $(u_{\sigma,a}, u'_{\sigma,a})$ in a compact subset of $(0, \infty) \times \mathbb{R}$. The standard continuation theorem for ODEs would then extend the solution past $x_{\min}$, a contradiction. $\square$

Remark 3.3. Lemma 3.2 only asserts existence on $(0, a]$. Since the right-hand side of (2.3) is smooth on $\{u > 0\}$ for all $x \in \mathbb{R}$, the terminal solutions, and the limit profile constructed below, may continue to negative values of x ; see Proposition 8.4. The sign argument for $\Psi_{\sigma,a}$ uses $x > 0$: at a zero of Ψ one has $\Psi' = xu''$, whose sign is that of x .

Proof of Theorem B: existence and estimates. On every compact interval $[\varepsilon, E] \Subset (0, \infty)$, Lemma 3.2 gives uniform C^1 bounds for $\{u_{\sigma,a} : a \geq E\}$ and a uniform positive lower bound for the functions. Since the right-hand side of (2.3) is smooth on $(0, \infty) \times (0, \infty) \times \mathbb{R}$, the ODE gives uniform C^k bounds on each compact interval. A diagonal Arzelà–Ascoli argument therefore produces a sequence $a_j \rightarrow \infty$ for which

$$u_{\sigma,a_j} \longrightarrow u_\sigma \quad \text{in } C_{\text{loc}}^\infty((0, \infty)).$$

The limit solves (2.3) and inherits (1.5)–(1.6).

Passing to the limit in (3.4) gives

$$(3.8) \quad u_\sigma(x) = \sigma x + \frac{n-r}{r} x \int_x^\infty \frac{1}{t^2} \int_t^\infty \frac{s(1 + (u'_\sigma(s))^2)}{u_\sigma(s)} \mathcal{E}_{u_\sigma}(t, s) ds dt.$$

We justify the passage to the limit by dominated convergence. Fix $x > 0$ and extend the integrand for the terminal problem by zero when $s > a$. Lemma 3.2 gives, for every $a \geq x$ and $s \geq x$,

$$u_{\sigma,a}(s) \geq \sigma s, \quad |u'_{\sigma,a}(s)| \leq P_x := \sigma + \frac{3(n-r)}{2r\kappa\sigma^r} x^{-r-1}.$$

For $x \leq t \leq s \leq a$, the exponent in the kernel satisfies

$$\begin{aligned} \kappa \int_t^s z m_{\sigma,a}(z)^{r-1} (1 + (u'_{\sigma,a}(z))^2) dz &\geq \kappa\sigma^{r-1} \int_t^s z^r dz \\ &\geq C_x (s^2 - t^2), \end{aligned}$$

where

$$C_x := \frac{\kappa\sigma^{r-1}x^{r-1}}{2} > 0.$$

Also

$$0 \leq \frac{s(1 + (u'_{\sigma,a}(s))^2)}{u_{\sigma,a}(s)} \leq D_x := \frac{1 + P_x^2}{\sigma}.$$

Thus the absolute value of the double-integral density is bounded, independently of a , by

$$D_x t^{-2} e^{-C_x(s^2 - t^2)} \mathbf{1}_{\{x \leq t \leq s\}}.$$

This function is integrable: since $s^2 - t^2 \geq t(s - t)$,

$$\int_x^\infty t^{-2} \int_t^\infty e^{-C_x(s^2 - t^2)} ds dt \leq \frac{1}{C_x} \int_x^\infty t^{-3} dt < \infty.$$

Local smooth convergence supplies pointwise convergence of the integrands. The dominated convergence theorem on the fixed region $\{(t, s) : x \leq t \leq s < \infty\}$ therefore proves (3.8); in particular, the majorant does not depend on the terminal radius.

Define

$$J(t) = \int_t^\infty \frac{s(1 + (u'_\sigma(s))^2)}{u_\sigma(s)} \exp\left(-\kappa \int_t^s z m_\sigma(z)^{r-1} (1 + (u'_\sigma(z))^2) dz\right) ds.$$

Differentiating (3.8) gives the exact identity

$$(3.9) \quad \Psi_\sigma(x) = x u'_\sigma(x) - u_\sigma(x) = -\frac{n-r}{r} J(x) < 0.$$

The comparison used in Lemma 3.2, now with base point x , yields

$$J(x) \leq \frac{1}{\kappa \sigma^r x^r},$$

which proves (1.7). Proposition 2.1 now shows that the curvature vector belongs to Γ_r at every point. Thus the elliptic solution generates a homothetically shrinking solution of the S_r -flow. $\square$

4. UNIQUENESS FOR A PRESCRIBED CONICAL SLOPE

Throughout this section, let $1 \leq r \leq n-1$ and set $a_0 = (n-r)/r > 0$. The argument follows the integral approach to uniqueness of Kleene–Møller [21, Section 2, Proposition 1, p. 3951], adapted to the additional dependence of the coefficients on the profile when $r > 1$. We first show that the convergence of $u(x)/x$ to a positive slope already forces the decay needed for uniqueness. Constants in the following lemma may depend on the solution; the explicit constants in Theorem B are not needed in this step.

Lemma 4.1. *Let $d \in \mathbb{R}$ and let $u \in C^2((d, \infty))$ be a positive solution of (2.3) such that $u(x)/x \rightarrow \sigma > 0$, and put $d_+ = \max\{d, 0\}$. Then $\Psi = xu' - u < 0$ and $u(x) > \sigma x$ on (d_+, ∞) . Moreover, $u' > 0$ for all sufficiently large x , and*

$$(4.1) \quad u - \sigma x = O(x^{-r}), \quad u' - \sigma = O(x^{-r-1}), \quad \Psi = O(x^{-r})$$

as $x \rightarrow \infty$.

Proof. Write $p = u'$ and define

$$\mathcal{A}(x) = \kappa x u^{r-1} (1 + p^2)^{(r+1)/2}, \quad b(x) = a_0 \frac{x(1 + p^2)}{u}.$$

Since $\Psi' = xu''$, the profile equation gives

$$(4.2) \quad \Psi' = \mathcal{A}\Psi + b.$$

Here $b > 0$ on (d_+, ∞) , and $u(x) \geq (\sigma/2)x$ for all sufficiently large x implies

$$\mathcal{A}(x) \geq c_0 x^r, \quad c_0 = \kappa(\sigma/2)^{r-1} > 0.$$

We first prove negativity, using only the conical asymptote. If $\Psi(x_0) \geq 0$ at some point $x_0 > d_+$, the integrating-factor formula for (4.2) implies $\Psi(x) > 0$ for $x > x_0$. Choosing $x_1 > x_0$ sufficiently large then gives

$$\Psi(x) \geq \Psi(x_1) \exp\left(\frac{c_0}{r+1}(x^{r+1} - x_1^{r+1})\right), \quad x \geq x_1.$$

Since $(u/x)' = \Psi/x^2$, this contradicts the finite limit of u/x . Thus $\Psi < 0$ on (d_+, ∞) , and u/x decreases strictly to σ there. In particular, $u > \sigma x$ and $p < u/x$ on (d_+, ∞) .

At any zero of p , the profile equation reads

$$p' = \frac{a_0}{u} - \kappa u^r.$$

Because $u(x) \rightarrow \infty$, choose $R > d_+$ so that every zero of p in $[R, \infty)$ satisfies $p' < 0$. Thus p cannot cross from negative to positive values there. More explicitly, if $p(x_0) < 0$ and p later became nonnegative, its first subsequent zero z would satisfy $p(z) = 0$ and $p'(z) \geq 0$, contrary to the choice of R . If $p(x_0) = 0$, then $p'(x_0) < 0$, so $p < 0$ immediately to the right and the same first-zero argument applies. Hence $p(x) \leq 0$ for every $x \geq x_0$. It follows that u is nonincreasing on $[x_0, \infty)$, contradicting $u(x)/x \rightarrow \sigma > 0$, which implies $u(x) \rightarrow \infty$. Therefore p has no nonpositive value on a sufficiently distant half-line. Hence, after enlarging R ,

$$0 < p(x) < \frac{u(x)}{x} \leq M, \quad x \geq R,$$

for some $M > 0$. It follows that $|\Psi(x)| = O(x)$ and $0 < b(x) \leq B_0$ on this half-line.

Integrating (4.2) from x to T gives

$$\Psi(x) = e^{-\int_x^T \mathcal{A}(t) dt} \Psi(T) - \int_x^T b(s) e^{-\int_x^s \mathcal{A}(t) dt} ds.$$

The first term tends to zero because $\Psi(T) = O(T)$ and $\mathcal{A}(t) \geq c_0 t^r$. The integral converges by the same lower bound and the boundedness of b . Hence

$$(4.3) \quad \Psi(x) = - \int_x^\infty b(s) \exp\left(-\int_x^s \mathcal{A}(t) dt\right) ds.$$

For $s \geq x \geq R$ we have $\mathcal{A}(s) \geq c_0 x^r$, and therefore

$$0 < -\Psi(x) \leq B_0 \int_x^\infty e^{-c_0 x^r (s-x)} ds = \frac{B_0}{c_0} x^{-r}.$$

Finally,

$$\frac{u(x)}{x} - \sigma = - \int_x^\infty \frac{\Psi(t)}{t^2} dt, \quad u' - \sigma = \frac{u - \sigma x + \Psi}{x}.$$

These identities give $0 < u - \sigma x \leq B_0 [c_0(r+1)]^{-1} x^{-r}$ and the asserted estimate for $u' - \sigma$. $\square$

Theorem 4.2. *Let $1 \leq r \leq n-1$, $\kappa > 0$, and $\sigma > 0$. Suppose that $u_i \in C^2((d_i, \infty))$, $d_i \in \mathbb{R}$, $i = 1, 2$, are positive solutions of (2.3) and*

$$\lim_{x \rightarrow \infty} \frac{u_1(x)}{x} = \lim_{x \rightarrow \infty} \frac{u_2(x)}{x} = \sigma.$$

Then $u_1 = u_2$ on (d_3, ∞) , $d_3 = \max\{d_1, d_2\}$.

Proof. Set

$$F(x, u, p) = \frac{a_0(1+p^2)}{u} + \kappa u^{r-1}(1+p^2)^{(r+1)/2}(xp - u).$$

Thus $u_i'' = F(x, u_i, u_i')$. For $0 \leq \theta \leq 1$, define

$$u_\theta = (1-\theta)u_2 + \theta u_1, \quad p_\theta = u_\theta',$$

and let

$$\alpha(x) = \int_0^1 F_u(x, u_\theta, p_\theta) d\theta, \quad \beta(x) = \int_0^1 F_p(x, u_\theta, p_\theta) d\theta.$$

The fundamental theorem of calculus implies that $w = u_1 - u_2$ satisfies $w'' = \alpha w + \beta w'$. The interpolation u_θ need not solve the profile equation. Nevertheless, Lemma 4.1 implies, uniformly for $\theta \in [0, 1]$,

$$(4.4) \quad u_\theta = \sigma x + O(x^{-r}), \quad p_\theta = \sigma + O(x^{-r-1}), \quad xp_\theta - u_\theta = O(x^{-r}).$$

The appropriate unknown is $y = w/x$. Direct substitution gives

$$(4.5) \quad y'' - P(x)y' = Q(x)y, \quad P = \beta - \frac{2}{x}, \quad Q = \alpha + \frac{\beta}{x}.$$

We verify the estimates

$$(4.6) \quad P(x) \geq c_* x^r, \quad |Q(x)| \leq C_* x^{-2}$$

on a sufficiently distant half-line, with $c_*, C_* > 0$. For this calculation write $W = (1 + p^2)^{1/2}$ and $\Psi = xp - u$. Then

$$\begin{aligned} F_u &= -\frac{a_0 W^2}{u^2} + \kappa(r-1)u^{r-2}W^{r+1}\Psi - \kappa u^{r-1}W^{r+1}, \\ F_p &= \frac{2a_0 p}{u} + \kappa x u^{r-1}W^{r+1} + \kappa(r+1)u^{r-1}pW^{r-1}\Psi. \end{aligned}$$

Using (4.4), we obtain

$$\beta(x) = \kappa \sigma^{r-1} (1 + \sigma^2)^{(r+1)/2} x^r + O(x^{-1}),$$

which yields the lower bound for P . More importantly, the leading terms in $F_u + F_p/x$ cancel exactly:

$$(4.7) \quad F_u + \frac{F_p}{x} = a_0 \left(\frac{2p}{xu} - \frac{W^2}{u^2} \right) + \kappa \Psi \left((r-1)u^{r-2}W^{r+1} + \frac{r+1}{x} u^{r-1}pW^{r-1} \right).$$

At $(u, p) = (u_\theta, p_\theta)$ the first term on the right-hand side is $O(x^{-2})$; in the second term the factor in parentheses is $O(x^{r-2})$ and $\Psi = O(x^{-r})$. Integration in θ gives the bound for Q . This calculation also applies to $r = 1$, when the term with factor $r - 1$ vanishes.

Lemma 4.1 gives

$$y = O(x^{-r-1}), \quad y' = O(x^{-r-2}).$$

Integrating (4.5) from x to T and letting $T \rightarrow \infty$, the terminal term $y'(T) \exp(-\int_x^T P)$ tends to zero by (4.6). We obtain

$$(4.8) \quad y'(x) = - \int_x^\infty Q(s)y(s) \exp\left(-\int_x^s P(t) dt\right) ds.$$

The integral is absolutely convergent because y is bounded and (4.6) holds.

Choose R so that (4.6) holds for $x \geq R$ and set $M_R = \sup_{s \geq R} |y(s)| < \infty$. For $x \geq R$,

$$\begin{aligned} |y'(x)| &\leq C_* M_R \int_x^\infty s^{-2} e^{-c_* x^r (s-x)} ds \\ &\leq \frac{C_*}{c_*} M_R x^{-r-2}. \end{aligned}$$

Since $y(x) \rightarrow 0$, a second integration gives

$$|y(x)| \leq \frac{C_*}{c_*(r+1)} M_R x^{-r-1}.$$

Taking the supremum over $x \geq R$, we conclude that

$$(4.9) \quad M_R \leq \frac{C_*}{c_*(r+1)} R^{-r-1} M_R.$$

Enlarging R so that the coefficient is less than one yields $M_R = 0$. Thus $u_1 = u_2$ on $[R, \infty)$. Since F is smooth for $u > 0$, uniqueness for the initial value problem extends this equality throughout the common interval of definition. $\square$

Theorem 4.2 is the uniqueness assertion of Theorem B: a positive solution on (d, ∞) with $u(x)/x \rightarrow \sigma$ agrees with u_σ on the common interval of definition.

Corollary 4.3. *For each $\sigma > 0$, the finite-radius profiles converge without passing to a subsequence:*

$$u_{\sigma,a} \longrightarrow u_\sigma \quad \text{in } C_{\text{loc}}^\infty((0, \infty)) \quad \text{as } a \rightarrow \infty.$$

In particular, the limit does not depend on the choice of the sequence a_j .

Proof. The bounds in Lemma 3.2 and the profile equation imply precompactness on every compact subinterval, as in the existence proof. Every sequence $a_j \rightarrow \infty$ has a subsequence converging locally smoothly to a positive solution with asymptotic slope σ . Theorem 4.2 identifies each such limit with u_σ . If convergence of the full family failed on some compact interval in some C^k norm, a sequence witnessing that failure would have a subsequence converging to u_σ , a contradiction. $\square$

Proposition 4.4. *If $0 < \sigma < \sigma'$, then*

$$(4.10) \quad u_\sigma(x) < u_{\sigma'}(x) \quad \text{for every } x \geq R(\sigma, \sigma') := \left(\frac{n-r}{2r\kappa\sigma^r(\sigma' - \sigma)} \right)^{1/(r+1)}.$$

In particular, the family is strictly ordered outside an explicitly controlled compact region for every fixed pair of slopes.

Proof. By Theorem B,

$$u_{\sigma'}(x) > \sigma'x, \quad u_\sigma(x) \leq \sigma x + \frac{n-r}{2r\kappa\sigma^r}x^{-r}.$$

Consequently,

$$u_{\sigma'}(x) - u_\sigma(x) > (\sigma' - \sigma)x - \frac{n-r}{2r\kappa\sigma^r}x^{-r},$$

and the right-hand side is nonnegative exactly when $x \geq R(\sigma, \sigma')$. $\square$

Continuity, and in fact real analyticity, of $\sigma \mapsto u_\sigma$ is proved in Section 6. The ordering in Proposition 4.4 does not extend to all $x > 0$; see Remark 9.1.

5. COMPLETE ASYMPTOTIC EXPANSION

Let u be a positive solution on (d, ∞) with $u(x)/x \rightarrow \sigma > 0$. Throughout this section set $a_0 = (n-r)/r$, $q = r+1$, and $W_0 = \sqrt{1 + \sigma^2}$. The symbol q is distinct from the quantity $m = u\sqrt{1 + (u')^2}$ used in the profile equation. All estimates below are taken as $x \rightarrow \infty$.

Proposition 5.1. *The angular defect satisfies*

$$(5.1) \quad \Psi(x) \sim -\frac{n-r}{r\kappa\sigma^r(1 + \sigma^2)^{(r-1)/2}}x^{-r}.$$

Thus, $u'' = o(x^{-1})$ and $\mu = o(x^{-1})$.

Proof. Use $\mathcal{A}$ and b from the proof of Lemma 4.1. That lemma gives

$$\frac{\mathcal{A}(x)}{x^r} \longrightarrow \mathcal{A}_\infty = \kappa\sigma^{r-1}W_0^{r+1}, \quad b(x) \longrightarrow b_\infty = \frac{a_0W_0^2}{\sigma}.$$

In (4.3), set $s = x + t/x^r$. Then

$$x^r \Psi(x) = - \int_0^\infty b(x + t/x^r) \exp \left(- \int_x^{x+t/x^r} \mathcal{A}(z) dz \right) dt.$$

For every fixed $t \geq 0$, the exponent tends to $-\mathcal{A}_\infty t$. Indeed, after setting $z = x + \tau/x^r$, its negative is $\int_0^t x^{-r} \mathcal{A}(x + \tau/x^r) d\tau$, whose integrand converges uniformly on bounded τ -intervals to $\mathcal{A}_\infty$. The bounds $\mathcal{A}(z) \geq \underline{c}z^r$ and $|b(z)| \leq C$ for large z dominate the integrand by $Ce^{-\underline{c}t}$. Dominated convergence therefore yields

$$\lim_{x \rightarrow \infty} x^r \Psi(x) = - \int_0^\infty b_\infty e^{-\mathcal{A}_\infty t} dt = -\frac{b_\infty}{\mathcal{A}_\infty},$$

which proves (5.1). It follows that

$$\frac{a_0}{u} + \kappa(u\sqrt{1 + (u')^2})^{r-1}\Psi = o(x^{-1}).$$

Since $u' \rightarrow \sigma$, the profile equation and (2.2) give the remaining conclusions. This proof does not require differentiating an asymptotic remainder. $\square$

We next prove that the expansion extends to all orders, with derivative bounds on the remainders. The argument also proves Theorem C for any positive solution with limiting slope σ .

Proof of Theorem C. Construction of the coefficients. Put $t = x^{-q}$ and write $u(x) = xv(t)$. Dots denote derivatives with respect to t . Since $t' = -qx^{-q-1} = -qt/x$, the chain rule first gives $u' = v + xv't' = v - qt\dot{v}$. Hence $\Psi = xu' - u = -qxt\dot{v}$, and a second differentiation gives $u'' = (qt/x)(r\dot{v} + qt\ddot{v})$ because $q-1=r$. Equivalently,

$$u' = v - qt\dot{v}, \quad \Psi = -qxt\dot{v}, \quad xu'' = qt(r\dot{v} + qt\ddot{v}).$$

Writing $W = (1 + (v - qt\dot{v})^2)^{1/2}$, the profile equation becomes

$$(5.2) \quad \mathcal{H}[v] := \kappa q v^{r-1} W^{r+1} \dot{v} + qt(r\dot{v} + qt\ddot{v}) - \frac{a_0 W^2}{v} = 0.$$

The expression is analytic near $t = 0$, $v = \sigma > 0$, for bounded $\dot{v}$ and $\ddot{v}$. Seek a formal series $v = \sigma + \sum_{j \geq 1} c_j t^j$. Its constant coefficient in (5.2) gives

$$\kappa q \sigma^{r-1} W_0^{r+1} c_1 = \frac{a_0 W_0^2}{\sigma},$$

which is (1.9).

The second coefficient is

$$(5.3) \quad c_2 = -\frac{rc_1^2}{2} \left[\frac{1}{\sigma} + \frac{\sigma}{1+\sigma^2} \left(1 - r + \frac{r(r+1)}{n-r} \right) \right].$$

To verify this formula, take the coefficient of t . Indeed, put $S = 1 + \sigma^2 = W_0^2$. From

$$v = \sigma + c_1 t + c_2 t^2 + O(t^3), \quad v - qt\dot{v} = \sigma - rc_1 t + O(t^2),$$

we obtain

$$\begin{aligned} v^{r-1} W^{r+1} &= \sigma^{r-1} S^{(r+1)/2} \left[1 + c_1 \left(\frac{r-1}{\sigma} - \frac{r(r+1)\sigma}{S} \right) t + O(t^2) \right], \\ \frac{W^2}{v} &= \frac{S}{\sigma} + c_1 \left(-2r - \frac{S}{\sigma^2} \right) t + O(t^2). \end{aligned}$$

Taking the coefficient of t in (5.2) yields

$$\begin{aligned} 0 &= \kappa q \sigma^{r-1} S^{(r+1)/2} \left[2c_2 + c_1^2 \left(\frac{r-1}{\sigma} - \frac{r(r+1)\sigma}{S} \right) \right] \\ &\quad + qrc_1 + a_0 c_1 \left(2r + \frac{S}{\sigma^2} \right). \end{aligned}$$

Using $\kappa q \sigma^{r-1} S^{(r+1)/2} c_1 = a_0 S/\sigma$ and $a_0 = (n-r)/r$, elementary simplification gives exactly (5.3).

For $j \geq 1$, the coefficient of t^{j-1} involving c_j is exactly $\kappa q j \sigma^{r-1} W_0^{r+1} c_j$. To see this, the new term in v is of order t^j , its contribution to $v - qt\dot{v}$ is also of order t^j , and its contribution to $qt(r\dot{v} + qt\ddot{v})$ is of order t^j . Only the factor $\dot{v}$ in the first term of (5.2) contributes at order t^{j-1} . Thus, with $v_{j-1}(t) = \sigma + \sum_{i=1}^{j-1} c_i t^i$, the precise recursion is

$$(5.4) \quad c_j = -\frac{[t^{j-1}] \mathcal{H}[v_{j-1}]}{\kappa q j \sigma^{r-1} W_0^{r+1}}.$$

Here $[t^{j-1}]$ denotes the Taylor coefficient at zero. The denominator is positive, so the recursion uniquely determines every coefficient. No logarithmic terms are needed.

For $N \geq 1$ set

$$U_N(x) = \sigma x + \sum_{j=1}^N c_j x^{1-jq}.$$

If F is the right-hand side of the profile equation, as defined in the proof of Theorem 4.2, then

$$(5.5) \quad E_N(x) := U_N'' - F(x, U_N, U_N') = O(x^{-1-Nq}).$$

Indeed, $x E_N = \mathcal{H}[v_N](x^{-q})$, and the recursion cancels all Taylor coefficients of $\mathcal{H}[v_N]$ through degree $N-1$. This constructs formal approximations only; the following estimate compares them with the actual solution.

Control of the remainder and its first derivative. For large x , $U_N > 0$. Define $e_N = u - U_N$ and $y_N = e_N/x$. Linearizing F along the segment joining (U_N, U_N') to (u, u') as in Theorem 4.2 gives

$$(5.6) \quad y_N'' - P_N y_N' = Q_N y_N - \frac{E_N}{x}.$$

Both endpoints of this segment satisfy the estimates in (4.4). Therefore the algebraic identity (4.7), which does not require the interpolating profiles to solve the equation, yields

$$(5.7) \quad P_N(x) \geq \underline{c}x^r, \quad |Q_N(x)| \leq Cx^{-2}$$

for all sufficiently large x . Constants may depend on N . By Lemma 4.1, initially $y_N = O(x^{-q})$ and $y'_N = O(x^{-q-1})$. Thus integration from infinity in (5.6) is justified and gives

$$(5.8) \quad y'_N(x) = - \int_x^\infty \left(Q_N(s)y_N(s) - \frac{E_N(s)}{s} \right) \exp \left(- \int_x^s P_N(z) dz \right) ds.$$

The exponential lower bound in (5.7) kills the terminal term and makes the integral absolutely convergent.

Let $M_R = \sup_{s \geq R} |y_N(s)|$. By (5.5)–(5.8), for $x \geq R$,

$$\begin{aligned} |y'_N(x)| &\leq C \int_x^\infty (M_R s^{-2} + s^{-2-Nq}) e^{-\underline{c}x^r(s-x)} ds \\ &\leq CM_R x^{-r-2} + Cx^{-r-2-Nq}. \end{aligned}$$

Integrating once more, using $y_N(\infty) = 0$ and $q = r + 1$, gives

$$|y_N(x)| \leq CM_R x^{-q} + Cx^{-(N+1)q}.$$

The constants here are independent of R once it is sufficiently large. Taking the supremum for $x \geq R$ and then increasing R if necessary,

$$M_R \leq CR^{-q}M_R + CR^{-(N+1)q} \implies M_R \leq C'R^{-(N+1)q}.$$

Since this holds for every sufficiently large R , it proves $y_N = O(x^{-(N+1)q})$. Substituting this pointwise estimate back into (5.8) gives $y'_N = O(x^{-1-(N+1)q})$. Hence

$$(5.9) \quad u - U_N = O(x^{1-(N+1)q}), \quad u' - U'_N = O(x^{-(N+1)q}).$$

Control after arbitrary differentiation. We supply this step separately, since differentiating a big- O estimate without further information is not valid. The smoothness of F for $u > 0$ and the bounds $u \asymp x$, $u' = O(1)$ first give $u'' = O(x^{-1})$. Repeated differentiation of the ODE then shows that, for each integer $\ell \geq 0$, some finite exponent B_ℓ satisfies

$$|u^{(\ell)}(x)| = O(x^{B_\ell}).$$

For completeness, every partial derivative of $F(x, u, p)$ has at most polynomial growth when $u \asymp x$ and p is bounded: the denominators are powers of u and $1 + p^2$, and the remaining factors are powers of x , u , and bounded functions of p . The chain rule therefore gives the assertion inductively from the estimates for u and u' .

For fixed k , the $2k$ th derivative of the linear term σx vanishes, and every remaining monomial in $U_M^{(2k)}$ has exponent $1 - jq - 2k \leq 1 - q - 2k < 0$. Hence, $|U_M^{(2k)}(x)| \leq C_{M,k}$ for $x \geq 1$: the constant may depend on M , but the polynomial exponent does not. Taking $B = \max\{B_{2k}, 0\}$ therefore gives, independently of M ,

$$\|(u - U_M)^{(2k)}\|_{L^\infty([x-1, x+1])} \leq C_M x^B.$$

We record the interpolation estimate used here. If $f \in C^{2k}(I)$ and $[x-1, x+1] \subset I$, then

$$|f^{(k)}(x)| \leq C_k \left(\|f\|_{L^\infty(I_x)} + \|f\|_{L^\infty(I_x)}^{1/2} \|f^{(2k)}\|_{L^\infty(I_x)}^{1/2} \right), \quad I_x = [x-1, x+1].$$

Indeed, Taylor interpolation at $2k$ fixed distinct nodes, rescaled into $[x-h, x+h]$ with $0 < h \leq 1$, yields

$$|f^{(k)}(x)| \leq C_k (h^{-k} \|f\|_\infty + h^k \|f^{(2k)}\|_\infty).$$

If $\|f^{(2k)}\|_\infty \leq \|f\|_\infty$, take $h = 1$. Otherwise take $h = (\|f\|_\infty / \|f^{(2k)}\|_\infty)^{1/(2k)}$. These two cases give the displayed inequality. This is the local one-dimensional Landau–Kolmogorov interpolation inequality; the argument above is included so that no differentiation of a big- O remainder is being assumed.

Applying it to $f = u - U_M$, using (5.9), gives

$$|(u - U_M)^{(k)}(x)| \leq C_{M,k} \left(x^{1-(M+1)q} + x^{(1-(M+1)q+B)/2} \right).$$

Given N and k , choose $M > N$ large enough that both exponents are at most $1 - (N + 1)q - k$. Since

$$u - U_N = (u - U_M) + \sum_{j=N+1}^M c_j x^{1-jq},$$

we obtain

$$(u - U_N)^{(k)} = O(x^{1-(N+1)q-k}).$$

Together with (5.9), this proves (1.8) for every N and k .

Finally, taking $N = 1$ and $k = 0, 1, 2$ yields

$$\begin{aligned} u &= \sigma x + c_1 x^{-r} + O(x^{-2r-1}), \\ u' &= \sigma - r c_1 x^{-r-1} + O(x^{-2r-2}), \\ u'' &= r(r+1) c_1 x^{-r-2} + O(x^{-2r-3}). \end{aligned}$$

Using $\mu = -u''(1 + (u')^2)^{-3/2}$ gives (1.10) and completes the proof. $\square$

Corollary 5.2. *The angular defect has the sharper expansion*

$$(5.10) \quad \Psi(x) = -\frac{n-r}{r\kappa\sigma^r(1+\sigma^2)^{(r-1)/2}} x^{-r} + O(x^{-2r-1}).$$

In particular, $u'' > 0$ and $\mu < 0$ for all sufficiently large x .

Proof. Use $\Psi = xu' - u$, the first two expansions above, and $c_1 > 0$. The signs follow from the expansion of u'' and (1.10). $\square$

Remark 5.3. The preceding result gives a complete classical asymptotic expansion, differentiable term by term, with exponent set $\{1 - j(r+1) : j \geq 0\}$ and without logarithmic terms. Here “complete” means precisely the remainder estimate (1.8) for every finite truncation and every derivative order; it does not assert convergence of the infinite formal series. The proof of the leading defect in Proposition 5.1 is independent of the construction of the higher-order coefficients.

6. ANALYTIC DEPENDENCE AND EXTERIOR FOLIATION

The first-order drift in the profile equation grows like x^r . Consequently, its inverse must not be treated as a uniformly bounded two-derivative gain in ordinary dyadically weighted Hölder spaces. We first use a graph norm, which requires only the weighted C^1 estimates supplied by the integral equation. The expansion proved in Section 5 then upgrades parameter analyticity to all weighted derivative orders.

For $R > 0$, let $\mathcal{C}_\delta^j(R)$ be the space of functions on $[R, \infty)$ with norm

$$\|z\|_{\mathcal{C}_\delta^j(R)} = \sum_{i=0}^j \sup_{x \geq R} x^{\delta+i} |z^{(i)}(x)|.$$

For $0 < \alpha < 1$, define

$$\|z\|_{\mathcal{X}_\delta^{k,\alpha}(R)} = \|z\|_{\mathcal{C}_\delta^k(R)} + \sup_{x \geq R} x^{\delta+k+\alpha} [z^{(k)}]_{C^{0,\alpha}([x, 2x])}.$$

All spaces below also have their complex-valued versions.

6.1. The linear inverse in a graph norm. Fix $\sigma_0 > 0$ and write

$$\alpha_0(x) = F_u(x, u_{\sigma_0}, u'_{\sigma_0}), \quad \beta_0(x) = F_p(x, u_{\sigma_0}, u'_{\sigma_0}), \quad L_0 z = z'' - \alpha_0 z - \beta_0 z'.$$

Here F is the right-hand side of (2.3), as defined in the proof of Theorem 4.2. Let

$$\mathcal{D}_0(R) = \{z \in C^2([R, \infty)) : z \in \mathcal{C}_r^1(R), L_0 z \in \mathcal{C}_1^0(R)\},$$

with the graph norm

$$\|z\|_{\mathcal{D}_0(R)} = \|z\|_{\mathcal{C}_r^1(R)} + \|L_0 z\|_{\mathcal{C}_1^0(R)}.$$

This is a Banach space: a Cauchy sequence converges in weighted C^1 , and the identity $z'' = L_0 z + \alpha_0 z + \beta_0 z'$ gives convergence of its second derivatives on every compact interval. No boundary condition is imposed at R ; the two conditions at infinity are contained in the weighted C^1 norm.

Lemma 6.1. *For sufficiently large R , the operator*

$$L_0 : \mathcal{D}_0(R) \longrightarrow \mathcal{C}_1^0(R)$$

is an isomorphism. Its weighted C^1 estimate can be chosen uniformly when σ_0 ranges over a compact subinterval of $(0, \infty)$.

Proof. Put $z = xy$. The equation $L_0 z = h$ becomes

$$(6.1) \quad y'' - P_0 y' = Q_0 y + \frac{h}{x}, \quad P_0 = \beta_0 - \frac{2}{x}, \quad Q_0 = \alpha_0 + \frac{\beta_0}{x}.$$

The calculation in (4.7) gives, for large R ,

$$(6.2) \quad P_0(x) \geq \underline{c} x^r, \quad |Q_0(x)| \leq C x^{-2}.$$

These constants are uniform on compact slope intervals, by (1.5)–(1.7). A decaying solution must satisfy

$$(6.3) \quad y'(x) = - \int_x^\infty \left(Q_0(s)y(s) + \frac{h(s)}{s} \right) e^{-\int_x^s P_0(t) dt} ds,$$

$$(6.4) \quad y(x) = - \int_x^\infty y'(t) dt.$$

The terminal term vanishes for $z \in \mathcal{D}_0(R)$. Conversely, a fixed point of these integrals is C^2 and solves (6.1).

Use $\|y\| = \sup_{x \geq R} x^{r+1} |y(x)|$ and $\|h\|_1 = \sup_{x \geq R} x |h(x)|$. The estimate $\int_x^s P_0 \geq \underline{c} x^r (s - x)$ gives

$$|y'(x)| \leq C \|h\|_1 x^{-r-2} + C \|y\| x^{-2r-3}, \quad |y(x)| \leq C \|h\|_1 x^{-r-1} + C \|y\| x^{-2r-2}.$$

The part depending on y therefore has operator norm at most CR^{-r-1} . Choose R so this is less than $1/2$ and apply the contraction principle in weighted C^0 . The resulting solution satisfies

$$(6.5) \quad \|z\|_{\mathcal{C}_r^1(R)} \leq C \|h\|_1.$$

Since $L_0 z = h$, it belongs to $\mathcal{D}_0(R)$ and its graph norm is bounded by $(C + 1)\|h\|_1$. Taking $h = 0$ proves injectivity. $\square$

6.2. Analyticity in weighted C^1 .

Lemma 6.2. *For R sufficiently large, the map*

$$\mathcal{G}(\sigma, v) = v'' - F(x, \sigma x + v, \sigma + v')$$

is real analytic from a neighborhood of $(\sigma_0, u_{\sigma_0} - \sigma_0 x)$ in $\mathbb{R} \times \mathcal{D}_0(R)$ to $\mathcal{C}_1^0(R)$. It has a holomorphic extension to a complex neighborhood.

Proof. Set $v_0 = u_{\sigma_0} - \sigma_0 x$. By Theorem C, $v_0 \in \mathcal{C}_r^1(R)$ and $v_0'' = O(x^{-r-2})$; also $\alpha_0 = O(x^{r-1})$ and $\beta_0 = O(x^r)$, so $v_0 \in \mathcal{D}_0(R)$. For v near v_0 and σ near σ_0 , write

$$a = \sigma + v/x, \quad p = \sigma + v', \quad d = xv' - v.$$

The functions a and p stay in fixed complex neighborhoods of the positive real number σ_0 , and $d = O(x^{-r})$. The function a^{-1} and the branch of $(1 + p^2)^{(r+1)/2}$ that is positive for real p are holomorphic on these neighborhoods. Moreover,

$$F(x, \sigma x + v, \sigma + v') = \frac{a_0(1 + p^2)}{xa} + \kappa x^{r-1} a^{r-1} (1 + p^2)^{(r+1)/2} d.$$

The weighted product estimates show that this expression is holomorphic with values in $\mathcal{C}_1^0(R)$. The second derivative causes no loss here: on the fixed graph domain,

$$v'' = L_0 v + \alpha_0 v + \beta_0 v'$$

is a bounded linear map to $\mathcal{C}_1^0(R)$. This proves the assertion. $\square$

At (σ_0, v_0) the derivative $D_v \mathcal{G}$ is L_0 . The analytic implicit-function theorem and Lemma 6.1 therefore give a holomorphic family v_σ in $\mathcal{D}_0(R)$ for σ in a complex disc about σ_0 . For real σ in this disc, $\sigma x + v_\sigma$ is positive and has limiting slope σ . Theorem 4.2 identifies it with u_σ . In particular, parameter analyticity is already established in $\mathcal{C}_r^1(R)$, without a two-derivative Schauder estimate.

6.3. Higher weighted derivative orders.

Lemma 6.3. *The holomorphic family just constructed is locally bounded in $\mathcal{X}_r^{k,\alpha}(R)$ for every $k \geq 0$ and $0 < \alpha < 1$.*

Proof. Work on a closed complex disc strictly inside the disc furnished by the implicit-function theorem, reducing the outer disc if necessary. The weighted C^1 bounds give uniformly

$$u_\sigma = \sigma x + O(x^{-r}), \quad u'_\sigma = \sigma + O(x^{-r-1}), \quad x u'_\sigma - u_\sigma = O(x^{-r}).$$

For each fixed N , the coefficients $c_j(\sigma)$ in (5.4) are holomorphic near σ_0 ; their denominators do not vanish there. The proof of Theorem C now applies uniformly to the complex family, with the following changes made explicit. Linearization along the segment joining u_σ to U_N uses the same holomorphic branch of F . The cancellation (4.7) still holds, and, on a sufficiently distant half-line (depending on N),

$$\operatorname{Re} P_N(x) \geq \underline{c} x^r, \quad |Q_N(x)| \leq C_N x^{-2}, \quad |E_N(x)| \leq C_N x^{-1-N(r+1)}.$$

Here $\underline{c} > 0$ follows by continuity from $\kappa \sigma_0^{r-1} (1 + \sigma_0^2)^{(r+1)/2} > 0$. Thus $|e^{-\int_x^s P_N}| \leq e^{-\underline{c} x^r (s-x)}$, and the two integral estimates for the remainder, including the absorption of M_R , are unchanged. They yield the C^0 and C^1 remainder bounds uniformly on the closed parameter disc.

Repeated differentiation of the ODE gives uniform polynomial bounds for higher x -derivatives: u_σ/x stays away from zero and u'_σ stays in the chosen holomorphic neighborhood. Applying the local interpolation argument in the expansion proof to real and imaginary parts gives its remainder estimates in every derivative order, uniformly on the parameter disc. In particular,

$$|\partial_x^j(u_\sigma - \sigma x)| \leq C_j x^{-r-j}.$$

On the remaining compact interval these estimates follow from the ODE and the uniform weighted C^1 bound. The bound for the next derivative implies, on each dyadic interval,

$$[\partial_x^k(u_\sigma - \sigma x)]_{C^{0,\alpha}([x, 2x])} \leq C_{k,\alpha} x^{-r-k-\alpha}.$$

This proves the claimed local boundedness in all the weighted spaces. $\square$

Theorem 6.4. *For every compact interval $I \Subset (0, \infty)$ there is $R_I > 0$ such that, for all $k \geq 0$ and $0 < \alpha < 1$, the map*

$$\sigma \mapsto u_\sigma - \sigma x \quad \text{with values in } \mathcal{X}_r^{k,\alpha}(R_I)$$

is real analytic on an open neighborhood of I . For $0 < \varepsilon < E < \infty$, the map

$$(0, \infty) \ni \sigma \mapsto u_\sigma \in C^{k,\alpha}([\varepsilon, E])$$

is also real analytic.

Proof. Fix σ_0 and the holomorphic family in the graph domain. On a parameter circle of radius d contained in its disc, Lemma 6.3 gives a bound $M_{k,\alpha}$ in $\mathcal{X}_r^{k,\alpha}(R)$. Pointwise Cauchy integration defines the Taylor coefficients $a_j(x)$ of $v_\sigma(x)$. Differentiation in x under this integral and the same estimate for difference quotients give

$$\|a_j\|_{\mathcal{X}_r^{k,\alpha}(R)} \leq M_{k,\alpha} d^{-j}.$$

The Taylor series therefore converges in this Banach space whenever $|\sigma - \sigma_0| < d$, and its pointwise sum is v_σ . This proves analyticity in every weighted derivative order. For σ in a compact interval I we take R_I as the maximum of the radii in a finite cover of I ; restriction to a larger half-line preserves all assertions.

Evaluation of u_σ and u'_σ at a fixed large radius is analytic. The first-order system $u' = p$, $p' = F(x, u, p)$ is real analytic for $u > 0$. On $[\varepsilon, R_I]$ the real solutions have a uniform positive lower bound. Analytic dependence on initial data therefore propagates the parameter analyticity to this compact interval. $\square$

Corollary 6.5. *For every compact interval $I \Subset (0, \infty)$ there is $R_I > 0$ such that*

$$\partial_\sigma u_\sigma(x) = x + O_I(x^{-r}) > 0 \quad (\sigma \in I, x \geq R_I).$$

For slopes in the interior of I , the rotational ends form a real-analytic foliation of their image in $\{x > R_I\}$. The same family extends analytically to a neighborhood of I .

Proof. Differentiate the analytic map with values in $\mathcal{C}_r^0(R_I)$. Its derivative is uniformly bounded on I , which gives the stated estimate and, after increasing R_I , the lower bound $x/2$. The ODE is analytic in both x and its initial data, so $(x, \sigma) \mapsto u_\sigma(x)$ is jointly real analytic. The map $(x, \sigma, \omega) \mapsto (x, u_\sigma(x)\omega)$ has nonzero Jacobian in the slope direction and is injective on the indicated region. The analytic inverse-function theorem gives a foliation of its image; no claim of coverage of the entire ambient exterior is made. $\square$

Remark 6.6. Analyticity and the exterior foliation are statements on distant exterior regions. They do not imply that $u_\sigma(x) < u_{\sigma'}(x)$ for every $x > 0$ when $\sigma < \sigma'$, and numerically this global ordering appears to fail in the computed examples; see Remark 9.1.

7. UNIQUENESS WITHOUT ROTATIONAL SYMMETRY

The uniqueness established above concerns rotational profiles. We now remove the symmetry assumption on the competing end, while retaining a circular asymptotic cone. The proof uses the equation to obtain decay of the normal graph and its difference from another solution, followed by Bernstein's continuation theorem at infinity [10]. The argument also applies to certain noncircular cones, as explained at the end of the section.

Throughout this section, $A = -dN$, S_r is unnormalized, and $\gamma = -c > 0$. Thus the geometric equation is

$$(7.1) \quad S_r(A) = -\gamma \langle X, N \rangle.$$

If the profile equation is written with coefficient κ , then $\gamma = \binom{n-1}{r-1} \kappa$.

Let

$$C = \{\rho\theta : \rho > 0, \theta \in \Gamma\}$$

be an oriented regular cone, where $\Gamma \subset \mathbb{S}^n$ is a smooth closed embedded hypersurface. Write

$$g_C = d\rho^2 + \rho^2 h, \quad A_C \partial_\rho = 0, \quad A_C|_{T\Gamma} = \rho^{-1} K,$$

where K is the shape operator of the link. As in (1.3), by a smoothly asymptotic normal graph over C we mean a parametrization

$$(7.2) \quad X_f(\rho, \theta) = \rho\theta + f(\rho, \theta)N_C(\theta), \quad |\nabla_C^j f| = o(\rho^{1-j}) \quad (j \geq 0).$$

Smooth conical convergence with multiplicity one gives this description after restricting the end. The orientation of the graph is chosen to converge to N_C .

We use $O_\infty(\rho^{-a})$ for a function or tensor whose j th covariant derivative has norm $O(\rho^{-a-j})$ for every j . Equivalently, all conormal derivatives, generated by $\rho\partial_\rho$ and smooth vector fields on the link, satisfy the corresponding weighted bounds.

We prove Theorem A below. The conclusion concerns equality outside a compact set and does not identify arbitrary interior extensions. No quantitative rate beyond (7.2) is assumed.

7.1. Normal graphs and automatic decay. Put

$$B = I - fA_C, \quad Z = B^{-1}\nabla_C f, \quad W_f = \sqrt{1 + |Z|^2}.$$

On a sufficiently distant part of the cone, B is invertible and

$$dX_f(U) = BU + df(U)N_C, \quad N_f = \frac{N_C - Z}{W_f},$$

$$g_f(U, V) = g_C(BU, BV) + df(U)df(V).$$

Since $B\partial_\rho = \partial_\rho$, we obtain the exact identity

$$\langle X_f, N_f \rangle = \frac{f - \rho f_\rho}{W_f}.$$

Define $\mathcal{E}(f) = W_f S_r(A_f)$. Equation (7.1) becomes

$$(7.3) \quad \gamma(\rho f_\rho - f) = \mathcal{E}(f).$$

For later estimates, the exact second fundamental form is

$$(7.4) \quad \begin{aligned} W_f b_f(U, V) &= b_C(U, V) - f g_C(A_C U, A_C V) + \nabla_C^2 f(U, V) \\ &\quad + df(U)g_C(A_C V, Z) + df(V)g_C(A_C U, Z) \\ &\quad + f g_C((\nabla_U A_C)V, Z). \end{aligned}$$

Indeed, after subtracting $dX_f(\nabla_U V)$ from $D_U dX_f(V)$, its normal component relative to the cone is

$$b_C(U, BV) + \nabla_C^2 f(U, V),$$

and its tangential component is

$$-df(U)A_C V - f(\nabla_U A_C)V - df(V)A_C U.$$

Taking the product with N_f proves (7.4). In particular, with our sign convention,

$$(7.5) \quad D\mathcal{E}(0)[v] = \text{tr}(T_{r-1}(A_C)\nabla_C^2 v) + \text{tr}(T_{r-1}(A_C)A_C^2)v,$$

where T_{r-1} is the Newton tensor.

Lemma 7.1. *For a regular cone C , every solution satisfying (7.2) obeys*

$$(7.6) \quad f = -\frac{S_r(K)}{\gamma(r+1)}\rho^{-r} + O_\infty(\rho^{-2r-1}).$$

If f_1, f_2 are two such solutions over the same oriented cone, then

$$(7.7) \quad f_1 - f_2 = O_\infty(\rho^{-M}) \quad \text{for every } M > 0.$$

Proof. Let $\mathcal{D}$ denote the algebra of conormal differential operators generated by $\rho\partial_\rho$ and by smooth vector fields on the link Γ . Thus, for a scalar function q , $q = O_\infty(\rho^{-\alpha})$ means that $|\mathcal{D}q| \leq C_{\mathcal{D}}\rho^{-\alpha}$ for every $\mathcal{D} \in \mathcal{D}$. This formulation is equivalent to the covariant-derivative definition of O_∞ on the cone.

The graph condition (7.2), together with the exact graph formulas, implies

$$\mathcal{E}(f) = O_\infty(\rho^{-r}).$$

Indeed, on every rescaled annulus $\{\rho \leq |X| \leq 2\rho\}$, the quantity $\rho^r \mathcal{E}(f)$ is a smooth function of θ and of the dimensionless graph jets

$$\frac{f}{\rho}, \quad \nabla_C f, \quad \rho \nabla_C^2 f,$$

and their conormal derivatives. These jets are bounded and converge to zero, with all conormal derivatives, by (7.2).

Dividing (7.3) by ρ^2 gives

$$\partial_\rho \left(\frac{f}{\rho} \right) = \frac{\mathcal{E}(f)}{\gamma\rho^2}.$$

Since $f/\rho \rightarrow 0$ as $\rho \rightarrow \infty$, integration from ρ to infinity yields

$$(7.8) \quad f(\rho, \theta) = -\frac{\rho}{\gamma} \int_{\rho}^{\infty} \frac{\mathcal{E}(f)(t, \theta)}{t^2} dt.$$

The integral is absolutely convergent because $\mathcal{E}(f) = O_{\infty}(\rho^{-r})$ and $r \geq 1$.

We record the elementary mapping property that will be used repeatedly: if $q = O_{\infty}(\rho^{-\alpha})$, with $\alpha > -1$, then

$$\mathcal{I}q(\rho, \theta) := \rho \int_{\rho}^{\infty} \frac{q(t, \theta)}{t^2} dt = O_{\infty}(\rho^{-\alpha}).$$

For angular conormal fields this follows by differentiating under the integral sign. For the radial conormal field one has the exact identity

$$\rho \partial_{\rho}(\mathcal{I}q) = \mathcal{I}q - q.$$

Iteration of this identity proves the assertion for arbitrary conormal derivatives. Applying it to (7.8) gives

$$f = O_{\infty}(\rho^{-r}).$$

We now improve this first estimate. On rescaled annuli, $\mathcal{E}(f)$ has the form

$$\mathcal{E}(f) = \rho^{-r} \mathcal{F} \left(\theta, \frac{f}{\rho}, \nabla_C f, \rho \nabla_C^2 f \right),$$

where $\mathcal{F}$ is smooth in its jet variables. The estimate just obtained implies that every dimensionless graph jet is $O_{\infty}(\rho^{-r-1})$. At the zero graph,

$$\mathcal{F}(\theta, 0, 0, 0) = S_r(K)(\theta).$$

Taylor expansion in the jet variables, uniformly on the compact link, therefore gives

$$\mathcal{E}(f) = \rho^{-r} S_r(K) + O_{\infty}(\rho^{-2r-1}).$$

Substituting this expansion into (7.8) and using

$$\rho \int_{\rho}^{\infty} t^{-r-2} dt = \frac{1}{r+1} \rho^{-r},$$

we obtain

$$f(\rho, \theta) = -\frac{S_r(K)(\theta)}{\gamma(r+1)} \rho^{-r} + O_{\infty}(\rho^{-2r-1}),$$

which is (7.6).

Let now f_1 and f_2 be two solutions over the same oriented cone, and set

$$w = f_1 - f_2, \quad f_{\tau} = f_2 + \tau w, \quad 0 \leq \tau \leq 1.$$

Subtracting their equations and applying the fundamental theorem of calculus in the graph variable gives the exact equation

$$(7.9) \quad \gamma(\rho w_{\rho} - w) = \mathcal{P}w, \quad \mathcal{P} = \int_0^1 D\mathcal{E}(f_{\tau}) d\tau.$$

In covariant form,

$$\mathcal{P}w = a^{ij} \nabla_{C,i} \nabla_{C,j} w + b^i \nabla_{C,i} w + dw,$$

where, with all conormal derivatives,

$$a = O_{\infty}(\rho^{1-r}), \quad b = O_{\infty}(\rho^{-r}), \quad d = O_{\infty}(\rho^{-r-1}).$$

These estimates follow from the smooth dependence of $\mathcal{E}$ on the dimensionless graph jets and from the uniform estimates for the segment $\{f_{\tau} : 0 \leq \tau \leq 1\}$.

So, for every $\alpha > 0$,

$$(7.10) \quad w = O_{\infty}(\rho^{-\alpha}) \quad \text{implies} \quad \mathcal{P}w = O_{\infty}(\rho^{-\alpha-r-1}).$$

Here the two covariant derivatives in the principal term contribute $\rho^{-\alpha-2}$, which, combined with $a = O_{\infty}(\rho^{1-r})$, gives precisely $\rho^{-\alpha-r-1}$. The first- and zeroth-order terms have the same decay.

As before, the condition $w/\rho \rightarrow 0$ allows us to integrate (7.9):

$$w(\rho, \theta) = -\frac{\rho}{\gamma} \int_{\rho}^{\infty} \frac{\mathcal{P}w(t, \theta)}{t^2} dt.$$

The mapping property of $\mathcal{I}$ and (7.10) therefore show that

$$w = O_\infty(\rho^{-\alpha}) \quad \text{implies} \quad w = O_\infty(\rho^{-\alpha-r-1}).$$

Initially, $w = O_\infty(\rho^{-r})$ because both f_1 and f_2 have that decay. In fact, the common leading term already gives $w = O_\infty(\rho^{-2r-1})$, but the weaker initial estimate suffices. Iterating the preceding implication yields

$$w = O_\infty(\rho^{-r-j(r+1)}) \quad \text{for every } j \geq 0.$$

Given any $M > 0$, choosing j sufficiently large proves

$$f_1 - f_2 = w = O_\infty(\rho^{-M}),$$

which is (7.7). $\square$

Notice that flatness at infinity alone does not imply equality. The remaining argument supplies the required continuation principle.

7.2. Bernstein's theorem on weakly conical ends. We recall the result that provides this principle. Following [10, Section 2, p. 1899], a triple $(E, g, \mathcal{R})$, where E is a smooth n -dimensional manifold with $n \geq 2$, g a C^1 Riemannian metric on E , and $\mathcal{R} : E \rightarrow (\mathcal{R}_E, \infty)$, with $\mathcal{R}_E \geq 1$, a proper unbounded C^2 function, is a *weakly conical end* if there is $\Lambda \geq 0$ such that

$$(7.11) \quad |\nabla_g \mathcal{R} - 1| \leq \frac{\Lambda}{\mathcal{R}^4} \leq \frac{1}{2}, \quad |\nabla_g^2 \mathcal{R}^2 - 2g| \leq \frac{\Lambda}{\mathcal{R}^2} \leq \frac{1}{2}.$$

Write $\partial_{\mathcal{R}} = \nabla_g \mathcal{R}$ and $\mathcal{L}_0 = \Delta_g - \frac{\mathcal{R}}{2} \partial_{\mathcal{R}}$.

Theorem 7.2 (Bernstein [10, Theorem 1.2, p. 1898]). *Let $(E, g, \mathcal{R})$ be a weakly conical end and let $u \in C^2(E)$ satisfy, for some constants $\varpi \in \mathbb{R}$ and $M \geq 0$,*

$$|(\mathcal{L}_0 + \varpi)u| \leq M\mathcal{R}^{-2}(|u| + |\nabla_g u|) \quad \text{and} \quad \int_E (|\nabla_g u|^2 + u^2) \mathcal{R}^{2-4\varpi} e^{-\mathcal{R}^2/4} < \infty.$$

Then there are constants R_0 and K_0 , depending on u , such that for every $R \geq R_0$

$$\int_{\{\mathcal{R} \geq R\}} \left(u^2 + \mathcal{R}^2 |\nabla_g u|^2 + \mathcal{R}^4 \left(\partial_{\mathcal{R}} u - \frac{2\varpi}{\mathcal{R}} u \right)^2 \right) \mathcal{R}^{-1-n-4\varpi} \leq \frac{K_0}{R^{n+4\varpi}} \int_{\{\mathcal{R}=R\}} u^2.$$

Moreover, u is asymptotically homogeneous of degree 2ϖ , with asymptotic trace $a \in H^1(L(E))$ on the link $L(E)$ of the asymptotic cone, and

$$\alpha^2 := \lim_{t \rightarrow \infty} t^{1-n-4\varpi} \int_{\{\mathcal{R}=t\}} u^2 = \int_{L(E)} a^2.$$

Finally, if $\mathcal{U}$ denotes the leading term of u , then for $R \geq R_0$

$$\int_{\{\mathcal{R} \geq R\}} \left(u^2 + \mathcal{R}^2 (|u - \mathcal{U}|^2 + |\nabla_g u|^2) + \mathcal{R}^4 \left(\partial_{\mathcal{R}} u - \frac{2\varpi}{\mathcal{R}} u \right)^2 \right) \mathcal{R}^{-2-n-4\varpi} \leq \frac{K_0 \alpha^2}{R^2}.$$

In particular, if $\alpha = 0$, the last estimate forces $u \equiv 0$ on $\{\mathcal{R} \geq R_0\}$. This is the unique continuation principle at infinity that we need; it is also the way in which Bernstein deduces the uniqueness of asymptotically conical self-shrinkers [10, Theorem 1.1, p. 1897, and Section 8, pp. 1920–1921].

7.3. The effective metric for a circular cone. For $C = \mathcal{C}_\sigma$, the principal curvatures of its link equal $k = \sigma^{-1}$. Define

$$a_R = \binom{n-1}{r-1} k^{r-1}, \quad a_T = \binom{n-2}{r-1} k^{r-1}, \quad \beta = \frac{r+1}{2}.$$

Notice that $2\beta = r+1 = q$, where q is the asymptotic reparametrization exponent introduced in Section 5. Thus the radial change of variables below is governed by the same homogeneity that determines the powers in the profile expansion. Both constants are positive in the range $1 \leq r \leq n-1$, and

$$\frac{a_T}{a_R} = \frac{n-r}{n-1}.$$

Equation (7.5) reads

$$(7.12) \quad \begin{aligned} D\mathcal{E}(0)[w] &= a_R \rho^{1-r} w_{\rho\rho} + (n-1)a_T \rho^{-r} w_\rho + a_T \rho^{-r-1} \Delta_h w \\ &\quad + (n-1)a_T k^2 \rho^{-r-1} w. \end{aligned}$$

In particular, the scaled principal symbol is positive definite. The same holds for $\mathcal{P}$ sufficiently far out.

We first record a tensorial identity for links with constant principal curvatures.

Lemma 7.3. *Let $\Gamma \subset \mathbb{S}^n$ be a smooth hypersurface whose principal curvatures, with their multiplicities, are constant, and let K be its shape operator. Then, for every polynomial p and every vector Y tangent to Γ ,*

$$\mathrm{tr}(p(K)\nabla_Y K) = 0, \quad \sum_i p(k_i) \langle (\nabla_{e_i} K)e_i, Y \rangle_h = 0,$$

where (e_i) is a local h -orthonormal frame of principal directions.

Proof. Let Φ be a polynomial with $\Phi' = p$. Then $\mathrm{tr}(p(K)\nabla_Y K) = Y(\mathrm{tr}\Phi(K))$, and $\mathrm{tr}\Phi(K) = \sum_i \Phi(k_i)$ is constant on Γ . For the second identity, $\nabla_{e_i} K$ is symmetric, and the Codazzi equation of Γ in the unit sphere gives $\langle (\nabla_{e_i} K)e_i, Y \rangle_h = \langle (\nabla_{e_i} K)Y, e_i \rangle_h = \langle (\nabla_Y K)e_i, e_i \rangle_h$. Summing with the weights $p(k_i)$ yields $\mathrm{tr}(p(K)\nabla_Y K) = 0$. $\square$

The next lemma makes precise in which sense the operator $\mathcal{P}$ of (7.9) is a small perturbation of the linearization at a radial function. Let $e_\rho = \partial_\rho$ and let $e_a = \rho^{-1}\epsilon_a$, where (ϵ_a) is a local h -orthonormal frame on the link, so that (e_ρ, e_a) is g_C -orthonormal. Write

$$\mathcal{P}w = \mathbf{a}^{\rho\rho}\nabla_{\rho\rho}^2 w + 2\mathbf{a}^{\rho a}\nabla_{\rho a}^2 w + \mathbf{a}^{ab}\nabla_{ab}^2 w + \mathbf{b}^\rho\nabla_\rho w + \mathbf{b}^a\nabla_a w + \mathbf{d}w$$

in this frame, with covariant derivatives of g_C .

Lemma 7.4. *Let C be the cone over a link Γ with constant principal curvatures, let $F = -\Theta\rho^{-r}$ with $\Theta = S_r(K)/(\gamma(r+1))$, and let $f_t = f_2 + tw$ as above. Then:*

- (i) *at F , the mixed coefficient and the angular first-order coefficient of $D\mathcal{E}(F)$ vanish. Its radial and zeroth-order coefficients depend only on ρ , while its angular principal block is a spectral function of K with scalar coefficients depending only on ρ . In a local principal frame its entries depend only on ρ ; this does not imply that K is parallel or that the entries are radial in an arbitrary orthonormal frame;*
- (ii) *each coefficient of $\mathcal{P}$ differs from the corresponding coefficient of $D\mathcal{E}(F)$ by $\rho^{1-r}O_\infty(\rho^{-2r-2})$ for the second-order coefficients, by $\rho^{-r}O_\infty(\rho^{-2r-2})$ for the first-order coefficients, and by $\rho^{-r-1}O_\infty(\rho^{-2r-2})$ for the zeroth-order coefficient.*

Proof. On a rescaled annulus $\{\rho \leq |X| \leq 2\rho\}$, ρ^{r-1} times the second-order coefficients of $D\mathcal{E}(f)$, ρ^r times its first-order coefficients, and ρ^{r+1} times its zeroth-order coefficient are smooth functions of θ and of the dimensionless jets $\mathcal{J}(f) = (f/\rho, \nabla_C f, \rho\nabla_C^2 f)$; indeed, $\mathcal{E}$ is a second-order operator in f , homogeneous under the dilations of the cone.

(i) For the radial function F , the graph metric, its shape operator and its Newton tensor are block diagonal with respect to the splitting into the radial direction and $T\Gamma$, and the angular blocks are spectral functions of K with coefficients depending only on ρ . Hence the mixed coefficient vanishes. Every term of the angular first-order coefficient contains either an angular derivative of F , which vanishes, or a contraction of ∇K with spectral functions of K , which vanishes by Lemma 7.3. For the circular cone this also follows from $O(n)$ -invariance: an $O(n)$ -invariant vector field on $\mathbb{S}^{n-1}$ vanishes.

(ii) By (7.6), $f_i = F + O_\infty(\rho^{-2r-1})$ for $i = 1, 2$, hence $\mathcal{J}(f_t) - \mathcal{J}(F) = O_\infty(\rho^{-2r-2})$ uniformly in $t \in [0, 1]$. Since the normalized coefficients are smooth functions of the jets, Taylor expansion around $\mathcal{J}(F)$ and integration in t give the assertion. $\square$

Lemma 7.5. *For the difference of two solutions over $\mathcal{C}_\sigma$, there are a smooth metric $\hat{g}$, a proper radial function $\mathcal{R}$, and constants ℓ, ϖ_* such that $v = \mathcal{R}^{\ell/2}w$ satisfies*

$$(7.13) \quad \left| (\Delta_{\hat{g}} + \varpi_*)v - \frac{\mathcal{R}}{2} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} v \rangle \right| \leq C\mathcal{R}^{-2}(|v| + |\nabla_{\hat{g}} v|).$$

Moreover,

$$(7.14) \quad |\nabla_g \mathcal{R}| = 1, \quad |\nabla_g^2 \mathcal{R}^2 - 2\hat{g}| \leq C\mathcal{R}^{-2},$$

and v and all its derivatives decay faster than every power of $\mathcal{R}$.

Proof. By Lemma 7.1, the two graph functions have the common leading term

$$f_i = F + O_\infty(\rho^{-2r-1}), \quad F = -\Theta\rho^{-r}, \quad \Theta = \frac{\binom{n-1}{r}k^r}{\gamma(r+1)}.$$

Therefore, for

$$f_t = f_2 + tw, \quad 0 \leq t \leq 1,$$

we have, uniformly in t ,

$$(7.15) \quad \frac{f_t - F}{\rho} = O_\infty(\rho^{-2r-2}).$$

For the radial graph determined by F , the meridian and angular principal curvatures are

$$\mu = \frac{F''}{(1 + F'^2)^{3/2}}, \quad \lambda = \frac{k + F'}{\rho(1 - kF/\rho)\sqrt{1 + F'^2}}.$$

Writing $W_F = \sqrt{1 + F'^2}$, we obtain

$$\mathcal{E}(F) = \binom{n-1}{r-1} \lambda^{r-1} \frac{F''}{W_F^2} + \binom{n-1}{r} W_F \lambda^r.$$

Recall that

$$a_R = \binom{n-1}{r-1} k^{r-1}, \quad a_T = \binom{n-2}{r-1} k^{r-1}, \quad \frac{a_T}{a_R} = \frac{n-r}{n-1}.$$

Since

$$F' = r\Theta\rho^{-r-1}, \quad \frac{kF}{\rho} = -k\Theta\rho^{-r-1},$$

expansion of the preceding formulas gives the coefficient of $w_{\rho\rho}$ in $D\mathcal{E}(F)$ as

$$(7.16) \quad a_R \rho^{1-r} (1 + \zeta \rho^{-r-1} + O_\infty(\rho^{-2r-2})), \quad \zeta = (r-1)\Theta(r/k - k).$$

The coefficient of w_ρ has leading term

$$(7.17) \quad (n-r)a_R \rho^{-r}.$$

Moreover, the angular principal part is

$$a_T \rho^{-r-1} \Delta_h$$

to leading order.

By Lemma 7.4, replacing $D\mathcal{E}(F)$ by

$$\mathcal{P} = \int_0^1 D\mathcal{E}(f_t) dt$$

changes each of its coefficients by a relative $O_\infty(\rho^{-2r-2}) = O_\infty(s^{-4})$, measured in the natural scale of that coefficient.

Set

$$\beta = \frac{r+1}{2}, \quad s = \rho^\beta, \quad A_0 = a_R \beta^2, \quad \eta = \frac{\gamma\beta}{A_0}.$$

Since

$$\rho \partial_\rho = \beta s \partial_s, \quad \rho^2 \partial_{\rho\rho} = \beta^2 s^2 \partial_{ss} + \beta(\beta-1)s \partial_s,$$

the equation

$$\mathcal{P}w - \gamma \rho w_\rho + \gamma w = 0$$

takes, in the variables (s, θ) , the form

$$(7.18) \quad A w_{ss} + 2A^a w_{sa} + A^{ab} w_{ab} + B w_s + B^a w_a + C w = 0.$$

Here $w_s = \partial_s w$, $w_{ss} = \partial_s^2 w$, $w_a = \partial_a w$, $w_{sa} = \partial_s \partial_a w$, and $w_{ab} = \nabla_a^h \nabla_b^h w$. Thus the angular second derivatives are covariant with respect to h . In ordinary coordinates the angular first-order coefficient is

$$B_{\text{coord}}^a = B^a - A^{bc} \Gamma_{bc}^a(h).$$

The connection term in this identity is retained when comparing the operator with a Laplace–Beltrami operator. The coefficients satisfy, with all derivatives measured on the conical scale,

$$(7.19) \quad A(s, \theta) = A_0 (1 + \zeta s^{-2} + O_\infty(s^{-4})),$$

$$(7.20) \quad B(s, \theta) = -\gamma \beta s + a_R \beta (\beta - 1 + n - r) s^{-1} + O_\infty(s^{-3}),$$

$$(7.21) \quad C(s, \theta) = \gamma + O_\infty(s^{-2}).$$

Furthermore,

$$(7.22) \quad A^{ab} = a_T s^{-2} (h^{ab} + s^{-2} Y^{ab} + O_\infty(s^{-4})),$$

$$(7.23) \quad A^a = O_\infty(s^{-5}) A_0,$$

where Y^{ab} is a constant multiple of h^{ab} and is independent of s . These expansions, as well as (7.19)–(7.21), follow from the radial computations above and Lemma 7.4. For instance, by Lemma 7.4(i) the mixed term of $\mathcal{P}$ is $2a^{\rho a} \nabla_{\rho a}^2 w$ with $a^{\rho a} = \rho^{1-r} O_\infty(\rho^{-2r-2})$; since $\nabla_{\rho a}^2 = \rho^{-1} \partial_\rho \epsilon_a$ up to lower-order terms and $\rho \partial_\rho = \beta s \partial_s$, its coefficient in (7.18) is $\rho^{-r-1} \beta s O_\infty(\rho^{-2r-2}) = O_\infty(s^{-5})$, which is (7.23). At the radial graph F the angular first-order coefficient vanishes after the angular Hessian has been written covariantly (Lemma 7.4(i)); this does not mean that the angular coordinate connection terms vanish. Its perturbation is $\mathbf{b}^a = \rho^{-r} O_\infty(\rho^{-2r-2})$, and the same conversion gives the coordinate coefficient $B^a = O_\infty(s^{-6})$. In particular,

$$(7.24) \quad |B^a|_{s^2 h} = O(s^{-3}).$$

For s sufficiently large, $A > 0$. Divide (7.18) by A and define a contravariant metric by

$$g^{ss} = 1, \quad g^{sa} = \frac{A^a}{A}, \quad g^{ab} = \frac{A^{ab}}{A}.$$

Strict ellipticity of the Newton tensor implies that this matrix is positive definite. Thus it is the inverse of a smooth Riemannian metric g . The principal symbol of the normalized operator is g^{ij} . In ordinary coordinates its second-order part is $g^{ij} \partial_{ij}$, and its angular first-order part includes $-(A^{bc}/A) \Gamma_{bc}^a(h) \partial_a$ in addition to B^a/A . At the radial graph F , the angular metric is a scalar multiple of h at each fixed s , so these connection terms agree exactly with the angular first-order terms of the corresponding Δ_g . The nonradial perturbation has relative size $O_\infty(s^{-4})$; its differentiated contribution to this comparison is of conical norm $O(s^{-5})$, which is smaller than the error retained below.

Define the limiting conical metric

$$g_* = ds^2 + h_{*ab} s^2 d\theta^a d\theta^b, \quad h_* = \frac{A_0}{a_T} h.$$

Equations (7.19)–(7.23), followed by inversion of the coefficient matrix, give

$$(7.25) \quad g^{ss} = 1,$$

$$(7.26) \quad g^{sa} = O_\infty(s^{-5}),$$

$$(7.27) \quad g^{ab} = s^{-2} \left(h_*^{ab} + s^{-2} \tilde{h}^{ab} + O_\infty(s^{-4}) \right),$$

where $\tilde{h}^{ab}$ is independent of s . Equivalently,

$$g = ds^2 + s^2 h_* + h_1 + O_\infty(s^{-2}),$$

up to the mixed terms in (7.26), where h_1 is a symmetric tensor on the link independent of s . In particular,

$$|g - g_*|_{g_*} = O(s^{-2})$$

with the corresponding conormal estimates in all orders.

Since $g^{ss} = 1$, we have the exact identity

$$|\nabla_g s|_g = 1.$$

The metric expansion also gives

$$\partial_s \log \sqrt{\det g} = \frac{n-1}{s} + O(s^{-3}),$$

and hence

$$(7.28) \quad \Delta_g s = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{is}) = \frac{n-1}{s} + O(s^{-3}).$$

For completeness, the Hessian identity

$$\nabla_g^2 s^2 = 2 ds \otimes ds + s \mathcal{L}_{\nabla_g s} g$$

also verifies the required tensor estimate directly. For the radial part $ds^2 + s^2 h_* + h_1$, subtraction of $2g$ leaves $-2h_1$ in the angular block, of conical norm $O(s^{-2})$. The differentiated remainder and the mixed terms have the same or smaller order. Hence

$$\nabla_g^2 s^2 = 2 ds \otimes ds + 2s \nabla_g^2 s = 2g + O_{g*}(s^{-2}).$$

Thus

$$(7.29) \quad |\nabla_g s|_g = 1, \quad |\nabla_g^2 s^2 - 2g|_g = O(s^{-2}).$$

Moreover, (7.26) implies

$$(7.30) \quad |\nabla_g s - \partial_s|_g = O(s^{-4}).$$

We next compare the normalized operator with Δ_g . In local coordinates,

$$\Delta_g w = g^{ij} \partial_{ij} w + \left(\frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij}) \right) \partial_j w.$$

Therefore the second-order terms cancel exactly when the normalized version of (7.18) is written in terms of Δ_g . By (7.28), the radial first-order term contributed by Δ_g is

$$\frac{n-1}{s} \partial_s w + O(s^{-3}) |\nabla_g w|.$$

On the other hand, dividing (7.20) by (7.19) gives

$$\begin{aligned} \frac{B}{A} &= \frac{-\gamma \beta s + a_R \beta (\beta - 1 + n - r) s^{-1} + O_\infty(s^{-3})}{A_0(1 + \zeta s^{-2} + O_\infty(s^{-4}))} \\ &= -\eta s + \left(\eta \zeta + \frac{\beta - 1 + n - r}{\beta} \right) s^{-1} + O_\infty(s^{-3}). \end{aligned}$$

Subtracting the radial first-order term of Δ_g , retaining the angular connection cancellation described above, and using (7.24), yields

$$(7.31) \quad \Delta_g w - \eta s \partial_s w + \frac{\ell}{s} \partial_s w + \frac{\gamma}{A_0} w = E_1 \cdot \nabla_g w + E_0 w,$$

where

$$|E_1|_g = O(s^{-3}), \quad |E_0| = O(s^{-2}),$$

and

$$(7.32) \quad \begin{aligned} \ell &= \eta \zeta + \frac{\beta - 1 + n - r}{\beta} - (n - 1) \\ &= \eta \zeta - \frac{n(r - 1)}{r + 1}. \end{aligned}$$

The term $\eta \zeta / s$ comes from the product of the leading drift $-\gamma \beta s$ with the correction $-\zeta s^{-2}$ in A^{-1} ; it is therefore part of the principal s^{-1} -drift and cannot be placed in the error.

Now put

$$\mathcal{R} = \sqrt{2\eta} s, \quad \hat{g} = 2\eta g.$$

Since constant rescaling gives

$$\Delta_{\hat{g}} = \frac{1}{2\eta} \Delta_g, \quad \nabla_{\hat{g}} \mathcal{R} \cdot \nabla = \frac{1}{\sqrt{2\eta}} \nabla_g s \cdot \nabla,$$

division of (7.31) by 2η , together with (7.30), gives

$$(7.33) \quad \left(\Delta_{\hat{g}} - \frac{\mathcal{R}}{2} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} \cdot \rangle + \frac{\ell}{\mathcal{R}} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} \cdot \rangle + \varpi_0 \right) w = \tilde{E}_1 \cdot \nabla_{\hat{g}} w + \tilde{E}_0 w,$$

where

$$\varpi_0 = \frac{\gamma}{2\eta A_0} = \frac{1}{r + 1}, \quad |\tilde{E}_1|_{\hat{g}} = O(\mathcal{R}^{-3}), \quad |\tilde{E}_0| = O(\mathcal{R}^{-2}).$$

Finally, set

$$w = \mathcal{R}^{-\ell/2} v.$$

Because $|\nabla_{\hat{g}} \mathcal{R}| = 1$, the first-order term produced by the Laplacian is

$$-\frac{\ell}{\mathcal{R}} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} v \rangle,$$

which cancels the $\ell/\mathcal{R}$ -drift in (7.33). The leading zeroth-order contribution from the Gaussian drift is

$$-\frac{\mathcal{R}}{2} \partial_{\mathcal{R}} (\mathcal{R}^{-\ell/2}) \mathcal{R}^{\ell/2} = \frac{\ell}{4},$$

whereas the remaining terms produced by the conjugation are $O(\mathcal{R}^{-2})v$. Therefore

$$\left| \left(\Delta_{\hat{g}} - \frac{\mathcal{R}}{2} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} \cdot \rangle + \varpi_* \right) v \right| \leq C \mathcal{R}^{-2} (|v| + |\nabla_{\hat{g}} v|),$$

with

$$\varpi_* = \frac{1}{r+1} + \frac{\ell}{4}.$$

This is (7.13).

Finally, (7.29) is invariant under the constant rescaling above and gives

$$|\nabla_{\hat{g}} \mathcal{R}|_{\hat{g}} = 1, \quad |\nabla_{\hat{g}}^2 \mathcal{R}^2 - 2\hat{g}|_{\hat{g}} \leq C \mathcal{R}^{-2},$$

which is (7.14). Since multiplication by the fixed power $\mathcal{R}^{\ell/2}$ preserves superalgebraic decay, (7.7) implies that v , together with all its derivatives, decays faster than every power of $\mathcal{R}$. $\square$

Proof of Theorem A. Let f_1, f_2 be the normal graph functions of the two ends over $\mathcal{C}_\sigma$, and let $w = f_1 - f_2$. By Lemma 7.5, there are a smooth metric $\hat{g}$ on a distant part E of the cone, a proper function $\mathcal{R}$, and constants ℓ, ϖ_* such that $v = \mathcal{R}^{\ell/2} w$ satisfies

$$(7.34) \quad |(\mathcal{L}_0 + \varpi_*)v| \leq C \mathcal{R}^{-2} (|v| + |\nabla_{\hat{g}} v|), \quad \mathcal{L}_0 = \Delta_{\hat{g}} - \frac{\mathcal{R}}{2} \partial_{\mathcal{R}},$$

where $\partial_{\mathcal{R}} = \nabla_{\hat{g}} \mathcal{R}$, and

$$|\nabla_{\hat{g}} \mathcal{R}|_{\hat{g}} = 1, \quad |\nabla_{\hat{g}}^2 \mathcal{R}^2 - 2\hat{g}|_{\hat{g}} \leq C \mathcal{R}^{-2}.$$

After increasing the inner radius, so that $C \mathcal{R}^{-2} \leq 1/2$, these estimates show that $(E, \hat{g}, \mathcal{R})$ is a weakly conical end in the sense of (7.11). Since $\hat{g}$ is uniformly equivalent to the conical metric $2\eta(ds^2 + s^2 h_*)$ and the link is compact, its level sets satisfy

$$(7.35) \quad \text{Vol}_{\hat{g}}\{\mathcal{R} = t\} \leq C t^{n-1}$$

for all sufficiently large t , and the volume of $\{t \leq \mathcal{R} \leq t+1\}$ is $O(t^{n-1})$.

By Lemma 7.5, v and all its derivatives decay faster than every power of $\mathcal{R}$: for every $N > 0$, $|v| + |\nabla_{\hat{g}} v| \leq C_N \mathcal{R}^{-N}$ on E . Hence

$$\int_E (v^2 + |\nabla_{\hat{g}} v|^2) \mathcal{R}^{2-4\varpi_*} e^{-\mathcal{R}^2/4} d\mu_{\hat{g}} < \infty,$$

and v satisfies the hypotheses of Theorem 7.2 with $\varpi = \varpi_*$. Moreover, by (7.35), for every $N > 0$,

$$t^{1-n-4\varpi_*} \int_{\{\mathcal{R}=t\}} v^2 d\sigma_{\hat{g}} \leq C_N^2 t^{1-n-4\varpi_*} t^{n-1} t^{-2N} = C_N^2 t^{-4\varpi_*-2N},$$

which tends to zero as $t \rightarrow \infty$ once $N > -2\varpi_*$. Thus the number α in Theorem 7.2 vanishes, and the last estimate of that theorem gives $v \equiv 0$ on $\{\mathcal{R} \geq R_0\}$. Since $\mathcal{R}^{\ell/2} > 0$, also $w = \mathcal{R}^{-\ell/2} v = 0$ there. Thus $f_1 = f_2$ on a sufficiently distant part of the cone, and the two ends coincide outside a compact set. The proof uses neither the rotational construction nor Theorems B and C; these enter only in Corollary 1.2. $\square$

7.4. Cones with constant principal curvatures on the link.

Proposition 7.6. *Let $\Gamma \subset \mathbb{S}^n$ be a smooth closed embedded oriented hypersurface with constant principal curvatures $k_1, \dots, k_{n-1}$ and constant multiplicities. Assume*

$$(0, k_1, \dots, k_{n-1}) \in \Gamma_r.$$

Two solutions of (7.1) that are smoothly asymptotic, with multiplicity one and compatible orientation, to the cone over Γ coincide outside a compact set.

Proof. Lemma 7.1 applies without change. Since the principal curvatures $k_1, \dots, k_{n-1}$ of the link are constant,

$$\Theta = \frac{S_r(K)}{\gamma(r+1)}$$

is constant on Γ . Hence the common leading term of the two normal graph functions is the radial function

$$F(\rho) = -\Theta\rho^{-r}.$$

Set

$$a_R = S_{r-1}(K), \quad T = T_{r-1}(K).$$

Because $(0, k_1, \dots, k_{n-1}) \in \Gamma_r$, the Newton tensor associated with the cone is positive definite. In particular,

$$a_R > 0, \quad T > 0.$$

Newton's identities also give

$$(7.36) \quad \text{tr}_h T = (n-r)S_{r-1}(K) = (n-r)a_R.$$

For the radial graph determined by F , the radial principal curvature and the angular principal curvatures are

$$\mu = \frac{F''}{(1+F'^2)^{3/2}}, \quad \lambda_i = \frac{k_i + F'}{\rho(1 - k_i F/\rho)\sqrt{1+F'^2}}.$$

The coefficient of the radial second derivative in $D\mathcal{E}(F)$ is

$$\frac{S_{r-1}(\lambda_1, \dots, \lambda_{n-1})}{1+F'^2}.$$

Using

$$F' = r\Theta\rho^{-r-1}, \quad \frac{F}{\rho} = -\Theta\rho^{-r-1},$$

we obtain

$$\lambda_i = \rho^{-1} (k_i + \Theta(r - k_i^2)\rho^{-r-1} + O(\rho^{-2r-2})).$$

Therefore

$$S_{r-1}(\lambda) = \rho^{1-r} \left[a_R + \Theta \sum_i S_{r-2}(k|i)(r - k_i^2)\rho^{-r-1} + O(\rho^{-2r-2}) \right],$$

where $k|i$ denotes the list obtained by omitting k_i . Since $F'^2 = O(\rho^{-2r-2})$, the radial coefficient becomes

$$a_R \rho^{1-r} (1 + \zeta \rho^{-r-1} + O_\infty(\rho^{-2r-2})),$$

where

$$(7.37) \quad \zeta = \frac{\Theta}{a_R} \sum_i S_{r-2}(k|i)(r - k_i^2) \quad (r \geq 2),$$

and $\zeta = 0$ when $r = 1$. Together with (7.36), this shows that all radial computations in the proof of Lemma 7.5 remain valid, with the present values of a_R and ζ .

It remains to analyze the angular part of the linearization. In an h -orthonormal frame diagonalizing K , the angular principal tensor of $D\mathcal{E}(F)$, after removing the common radial factor ρ^{-2} , is diagonal with eigenvalues

$$(7.38) \quad q_i(\rho) = \frac{S_{r-1}(\mu, \lambda|i)}{(1 - k_i F/\rho)^2}.$$

Denote this symmetric tensor by $\mathbf{Q}_\rho$.

For each fixed ρ , the value $q_i(\rho)$ depends on the principal direction only through the corresponding eigenvalue k_i . Hence $\mathbf{Q}_\rho$ is a spectral function of K . If $\nu_1, \dots, \nu_d$ are the

distinct eigenvalues of K , polynomial interpolation gives a polynomial p_ρ , of degree at most $d - 1$, such that

$$\mathbf{Q}_\rho = p_\rho(K).$$

The interpolation coefficients depend on ρ and on the constant numbers $\nu_1, \dots, \nu_d$, but not on the point of Γ . By the Cayley–Hamilton and Newton identities, $p_\rho(K)$ can be written as

$$\mathbf{Q}_\rho = \sum_{j=0}^{n-2} c_j(\rho) T_j(K),$$

where every coefficient $c_j(\rho)$ is constant on the link.

Each Newton tensor $T_j(K)$ is divergence-free with respect to h . Since the eigenvalues of K , and hence all $S_j(K)$, are constant, we conclude that

$$(7.39) \quad \operatorname{div}_h \mathbf{Q}_\rho = 0.$$

Moreover, the eigenvalues $q_i(\rho)$ are constant on Γ . Thus

$$(7.40) \quad \det_h \mathbf{Q}_\rho = \prod_{i=1}^{n-1} q_i(\rho)$$

is constant on the link for every fixed ρ . Moreover, strict admissibility and convergence to the conical coefficients imply that $\mathbf{Q}_\rho$ is positive definite for all sufficiently large ρ .

Define the radius-dependent angular metric h_ρ by

$$h_\rho^{-1} = \mathbf{Q}_\rho, \quad \text{that is,} \quad h_\rho^{ab} = \mathbf{Q}_\rho^{ab}.$$

By (7.40), the density ratio

$$\frac{d\mu_{h_\rho}}{d\mu_h} = (\det_h \mathbf{Q}_\rho)^{-1/2}$$

is constant on the link. Combining this fact with $\operatorname{div}_h \mathbf{Q}_\rho = 0$, we obtain, for every smooth function u on the link,

$$(7.41) \quad \begin{aligned} \Delta_{h_\rho} u &= \frac{1}{\sqrt{\det h_\rho}} \partial_a \left(\sqrt{\det h_\rho} h_\rho^{ab} \partial_b u \right) \\ &= \frac{1}{\sqrt{\det h}} \partial_a \left(\sqrt{\det h} \mathbf{Q}_\rho^{ab} \partial_b u \right) \\ &= \mathbf{Q}_\rho^{ab} \nabla_a^h \nabla_b^h u. \end{aligned}$$

Thus the angular principal part of the linearization at F is exactly a Laplace–Beltrami operator and introduces no additional angular first-order term.

After the radial change of variables and division by the radial principal coefficient, we denote the limiting normalized angular tensor by $\mathbf{Q}_\infty$; explicitly,

$$\mathbf{Q}_\infty^{ab} = \frac{1}{a_R \beta^2} T^{ab}, \quad \beta = \frac{r+1}{2}.$$

We therefore define the limiting link metric h_* by

$$(7.42) \quad h_*^{ab} = \frac{1}{a_R \beta^2} T^{ab}.$$

Since $T > 0$, this defines a smooth Riemannian metric on the link. Moreover, the eigenvalues of T are constant, so $\det_h T$ is constant. As above,

$$\Delta_{h_*} u = \frac{1}{a_R \beta^2} T^{ab} \nabla_a^h \nabla_b^h u.$$

We next verify that there are no further angular first-order terms in the linearization at the radial graph F . Since F depends only on ρ , every term containing an angular derivative of F vanishes. The only remaining possible contributions involving ∇K are contractions of the form $\operatorname{tr}(P(K) \nabla_Y K)$ and $\sum_i P(k_i) \langle (\nabla_{e_i} K) e_i, Y \rangle_h$, with polynomial weights representing the spectral functions of the constant eigenvalues; both vanish by Lemma 7.3. The radial-angular principal coefficients vanish at F because the graph metric, its shape operator, and its Newton tensor are block diagonal in the radial-angular splitting. The algebraic angular gradient terms contain no angular derivative of F . Thus these are tensorial cancellations; no assumption that $\nabla K = 0$ is being made. Hence

the angular part of the linearization at F is precisely the Laplacian associated with the radius-dependent effective metric.

After setting

$$s = \rho^\beta, \quad A_0 = a_R \beta^2, \quad \eta = \frac{\gamma \beta}{A_0},$$

the effective metric determined by the full principal symbol has the expansion

$$(7.43) \quad g = ds^2 + s^2 h_* + h_1 + O_\infty(s^{-2}),$$

up to mixed terms of relative order $O_\infty(s^{-4})$, where h_1 is independent of s . The angular connection cancellation follows from (7.41); one does not discard coordinate first-order terms. Indeed, the h_1 -term is produced by the first correction of the radial approximation F , while

$$\frac{f_t - F}{\rho} = O_\infty(\rho^{-2r-2}) = O_\infty(s^{-4})$$

shows that replacing F by f_t contributes only relative $O_\infty(s^{-4})$ -errors.

It follows that

$$|\nabla_g s|_g = 1, \quad |\nabla_g^2 s^2 - 2g|_g = O(s^{-2}), \quad \Delta_g s = \frac{n-1}{s} + O(s^{-3}).$$

Exactly as in Lemma 7.5, the normalized equation for $w = f_1 - f_2$ becomes

$$\Delta_g w - \eta s \partial_s w + \frac{\ell}{s} \partial_s w + \frac{\gamma}{A_0} w = E_1 \cdot \nabla_g w + E_0 w,$$

where

$$|E_1|_g = O(s^{-3}), \quad |E_0| = O(s^{-2}),$$

and

$$\ell = \eta \zeta + \frac{\beta - 1 + n - r}{\beta} - (n - 1) = \eta \zeta - \frac{n(r-1)}{r+1}.$$

Finally, put

$$\mathcal{R} = \sqrt{2\eta} s, \quad \hat{g} = 2\eta g, \quad v = \mathcal{R}^{\ell/2} w.$$

The conjugation by $\mathcal{R}^{\ell/2}$ cancels the $\ell/\mathcal{R}$ -drift and gives

$$\left| \left(\Delta_{\hat{g}} - \frac{\mathcal{R}}{2} \langle \nabla_{\hat{g}} \mathcal{R}, \nabla_{\hat{g}} \cdot \rangle + \varpi_* \right) v \right| \leq C \mathcal{R}^{-2} (|v| + |\nabla_{\hat{g}} v|),$$

with

$$\varpi_* = \frac{1}{r+1} + \frac{\ell}{4}.$$

Furthermore,

$$|\nabla_{\hat{g}} \mathcal{R}|_{\hat{g}} = 1, \quad |\nabla_{\hat{g}}^2 \mathcal{R}^2 - 2\hat{g}|_{\hat{g}} \leq C \mathcal{R}^{-2}.$$

Thus $(E, \hat{g}, \mathcal{R})$ is a weakly conical end. By Lemma 7.1, v and all its derivatives decay faster than every power of $\mathcal{R}$.

The proof of Theorem A now applies verbatim: the asymptotic trace of v given by Theorem 7.2 vanishes because of the superalgebraic decay, and therefore $v = 0$ on a sufficiently distant end. Hence $w = 0$ there, and the two hypersurface ends coincide outside a compact set. $\square$

Remark 7.7. The proposition includes noncircular cones. For $1 \leq r \leq n-2$, consider

$$\Gamma = \mathbb{S}^{n-2}(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^n, \quad a^2 + b^2 = 1, \quad k = b/a.$$

With a suitable orientation its principal curvatures are k , with multiplicity $p = n-2$, and $-1/k$, with multiplicity one. Hence

$$S_j(0, k, \dots, k, -1/k) = \binom{p}{j} k^j - \binom{p}{j-1} k^{j-2} > 0 \quad (1 \leq j \leq r)$$

whenever $k^2 > \frac{r}{n-1-r}$. Proposition 7.6 then gives uniqueness of S_r -shrinking ends asymptotic to the cone over Γ , provided such ends exist. For $r = 1$ this is contained in Wang's theorem; for $r \geq 2$ the existence of such ends is an open problem.

Remark 7.8. For an arbitrary regular cone, Lemma 7.1 still gives superalgebraic decay of the difference. The continuation argument above, however, does not apply directly. Even if the cone is strictly r -admissible, normalization of the radial principal coefficient produces the drift

$$-\frac{\gamma}{\beta S_{r-1}(K(\theta))} s \partial_s,$$

whose coefficient may depend on the angular variable. This is a principal term, not an error allowed in (7.13).

Further angular dependence may occur in the coefficient ℓ of the s^{-1} drift. Conjugation by a power with an angularly varying exponent introduces logarithmic derivatives and does not supply the required estimate. Finally, if the limiting cone lies on the boundary of Γ_r , positivity of the scaled principal symbol must itself be addressed. Thus uniqueness for general regular cones remains outside the scope of this argument.

8. GEOMETRIC CONSEQUENCES

Proposition 8.1. *For $R > 0$, set $u_{\sigma,R}(s) = R^{-1}u_{\sigma}(Rs)$. For every $0 < a < A < \infty$ and integer $k \geq 0$,*

$$\|u_{\sigma,R} - \sigma(\cdot)\|_{C^k([a,A])} = O(R^{-(r+1)}).$$

Furthermore,

$$\lim_{x \rightarrow \infty} x\lambda(x) = \frac{1}{\sigma\sqrt{1+\sigma^2}}, \quad |A_{\Sigma_{\sigma}}(x)| = O(x^{-1}).$$

Proof. For each $j \geq 0$, differentiation of the rescaled profile contributes R^{j-1} to the j th derivative. The differentiated expansion in Theorem C, uniformly for $s \in [a, A]$, therefore gives the asserted C^k estimate. In particular, the end is smoothly asymptotic to its cone with multiplicity one. Since

$$\lambda = \frac{1}{u_{\sigma}\sqrt{1+(u'_{\sigma})^2}},$$

the asserted limit follows from $u_{\sigma}/x \rightarrow \sigma$ and $u'_{\sigma} \rightarrow \sigma$. Finally, set

$$\Phi_{\sigma} = \frac{n-r}{ru_{\sigma}} + \kappa m_{\sigma}^{r-1} \Psi_{\sigma}.$$

Theorem B gives $\Phi_{\sigma} = O(x^{-1})$. Since $\mu = -\frac{\Phi_{\sigma}}{\sqrt{1+(u'_{\sigma})^2}}$, we obtain $\mu = O(x^{-1})$ and hence $|A| = O(x^{-1})$. $\square$

The expansion also determines the first correction to the transversal principal curvature.

Proposition 8.2. *As $x \rightarrow \infty$,*

$$(8.1) \quad \lambda(x) = \frac{1}{\sigma\sqrt{1+\sigma^2}}x^{-1} - \frac{c_1(1+(1-r)\sigma^2)}{\sigma^2(1+\sigma^2)^{3/2}}x^{-r-2} + O(x^{-2r-3}).$$

Consequently,

$$|A_{\Sigma_{\sigma}}(x)| = \frac{\sqrt{n-1}}{\sigma\sqrt{1+\sigma^2}}x^{-1} + O(x^{-r-2}).$$

Proof. Theorem C, with derivatives through order two, gives

$$u = \sigma x + c_1 x^{-r} + O(x^{-2r-1}), \quad u' = \sigma - r c_1 x^{-r-1} + O(x^{-2r-2}).$$

Therefore

$$\sqrt{1+(u')^2} = \sqrt{1+\sigma^2} \left(1 - \frac{r\sigma c_1}{1+\sigma^2} x^{-r-1} + O(x^{-2r-2}) \right),$$

and expansion of $\lambda = (u\sqrt{1+(u')^2})^{-1}$ proves (8.1). Finally, $|A|^2 = (n-1)\lambda^2 + \mu^2$ and $\mu = O(x^{-r-2})$ by (1.10); taking the positive square root gives the last assertion. $\square$

Proposition 8.3. *Let $\omega_{n-1} = |\mathbb{S}^{n-1}|$. Then*

$$\text{Vol}_n(\Sigma_\sigma \cap \{b \leq x \leq R\}) = \frac{\omega_{n-1}}{n} \sigma^{n-1} \sqrt{1 + \sigma^2} R^n + o(R^n).$$

Moreover, for every $b > 0$,

$$\int_{\Sigma_\sigma \cap \{x \geq b\}} |A|^n d\mu = \infty.$$

More precisely,

$$\int_{\Sigma_\sigma \cap \{b \leq x \leq R\}} |A|^n d\mu = \frac{\omega_{n-1}(n-1)^{n/2}}{\sigma(1+\sigma^2)^{(n-1)/2}} \log R + O(1).$$

Proof. The area form is

$$d\mu = u_\sigma^{n-1} \sqrt{1 + (u'_\sigma)^2} dx d\omega.$$

The volume asymptotics follow from $u_\sigma/x \rightarrow \sigma$ and $u'_\sigma \rightarrow \sigma$. By Proposition 8.2 and Theorem C, the curvature density satisfies

$$|A|^n u_\sigma^{n-1} \sqrt{1 + (u'_\sigma)^2} = \frac{(n-1)^{n/2}}{\sigma(1+\sigma^2)^{(n-1)/2}} x^{-1} + O(x^{-r-2}).$$

The remainder is integrable at infinity. Integration in x and over $\mathbb{S}^{n-1}$ proves the logarithmic asymptotic formula and the divergence. $\square$

We finally record what can be said about the profile away from infinity. Since the right-hand side of (2.3) is real analytic on $\{u > 0\}$ for every $x \in \mathbb{R}$, the point $x = 0$ plays no special role for the equation.

Proposition 8.4. *The profile u_σ extends uniquely to a maximal positive solution of (2.3) on an interval (x_σ, ∞) with $x_\sigma \in [-\infty, 0]$. If $x_\sigma > -\infty$, then $(u_\sigma(x), u'_\sigma(x))$ leaves every compact subset of $(0, \infty) \times \mathbb{R}$ as $x \downarrow x_\sigma$. Moreover:*

- (i) *if $x_\sigma < 0$, then $u_\sigma(0) > 0$, $\Psi_\sigma < 0$ on $[0, \infty)$, and $\Sigma_\sigma \cap \{x \geq 0\}$ is r -admissible;*
- (ii) *if $x_\sigma = 0$, then the limit $L_\sigma = \lim_{x \downarrow 0} u_\sigma(x)/x$ exists in $(\sigma, \infty]$; if $L_\sigma < \infty$, then $u_\sigma(x) \rightarrow 0$ and $u_\sigma(\rho s)/\rho \rightarrow L_\sigma s$ locally uniformly for $s > 0$ as $\rho \downarrow 0$.*

Proof. The existence of a maximal solution and the behavior at a finite endpoint are the standard continuation theorem for the first-order system $u' = p$, $p' = F(x, u, p)$, whose right-hand side is smooth on $\mathbb{R} \times (0, \infty) \times \mathbb{R}$. Since u_σ is defined on $(0, \infty)$, we have $x_\sigma \leq 0$.

(i) If $x_\sigma < 0$, then $u_\sigma(0) > 0$ and $\Psi_\sigma(0) = -u_\sigma(0) < 0$, while $\Psi_\sigma < 0$ on $(0, \infty)$ by (3.9). Since $\lambda > 0$ on a graph, Proposition 2.1 gives admissibility.

(ii) By (3.9), $(u_\sigma/x)' = \Psi_\sigma/x^2 < 0$ on $(0, \infty)$, so u_σ/x has a limit in $(\sigma, \infty]$ as $x \downarrow 0$. If $L_\sigma < \infty$, then $u_\sigma(x) = x(u_\sigma(x)/x) \rightarrow 0$, and for every compact interval $[a, A] \Subset (0, \infty)$,

$$\sup_{s \in [a, A]} \left| \frac{u_\sigma(\rho s)}{\rho} - L_\sigma s \right| \leq A \sup_{y \in [\rho a, \rho A]} \left| \frac{u_\sigma(y)}{y} - L_\sigma \right| \rightarrow 0. \quad \square$$

Proposition 8.5 (Obstruction at a vertical tangent). *Let $2 \leq r \leq n-1$ and $c \neq 0$. On a C^2 regular rotational immersion satisfying $S_r = c\langle X, N \rangle$, a vertical meridian tangent at positive radius can occur only in the plane $x = 0$. More precisely, suppose that a graphical solution approaches a finite point (x_*, y_*) with $x_* \neq 0$, $y_* > 0$, and $|u'| \rightarrow \infty$. Then it has no C^2 regular continuation as a solution of the same equation through that point. With the graphical orientation,*

$$(8.2) \quad \lambda \rightarrow 0, \quad \mu \lambda^{r-1} \rightarrow \frac{c x_* \epsilon}{\binom{n-1}{r-1}} \neq 0, \quad \epsilon = \lim \frac{u'}{\sqrt{1 + (u')^2}} \in \{-1, 1\}.$$

In particular, the meridian curvature diverges in absolute value.

Proof. Parametrize a regular meridian by arclength τ , so that $\dot{x}^2 + \dot{y}^2 = 1$, and choose $N = (\dot{y}, -\dot{x})$. At a vertical tangent, $\dot{x} = 0$ and $\dot{y} = \pm 1$. Hence $\lambda = \dot{x}/y = 0$ and $\langle X, N \rangle = \pm x$. The meridian curvature μ is finite on a C^2 regular immersion. Since

$$S_r = \binom{n-1}{r-1} \mu \lambda^{r-1} + \binom{n-1}{r} \lambda^r,$$

we have $S_r = 0$ for $r \geq 2$, and the soliton equation forces $x = 0$.

For the graphical limit in the statement, $\lambda = (u\sqrt{1+(u')^2})^{-1} \rightarrow 0$. Continuity of u' and $|u'| \rightarrow \infty$ imply that its sign is eventually constant, so $u'/\sqrt{1+(u')^2} \rightarrow \epsilon$ and

$$\langle X, N \rangle = \frac{xu' - u}{\sqrt{1+(u')^2}} \rightarrow x_*\epsilon.$$

The formula for S_r gives (8.2). Thus $|\mu|$ grows like a nonzero constant times λ^{1-r} , excluding a C^2 regular continuation. $\square$

Remark 8.6. (Numerical.) In the computed cases the profiles cross $x = 0$ at positive height and become steep near a negative left endpoint x_σ , at a positive radius; for $r \geq 2$, Proposition 8.5 then excludes a C^2 regular continuation. The angular defect can become positive before x_σ is reached, so r -admissibility may be lost within the graphical region. Approximate values:

n	r	σ	$u_\sigma(0)$	x_σ	$\{x > x_\sigma : \Psi_\sigma > 0\}$
3	1	1	1.187	-1.68	$(-1.68, -1.43)$
3	2	1	0.456	-1.37	$(-1.37, -0.93)$
4	2	0.5	0.937	-2.48	$\emptyset$
5	3	2	0.333	-0.81	$(-0.81, -0.49)$

Here $\kappa = 1$, and the computations are as in Figure 1.

For each $b \in \mathbb{R}$ with $b > x_\sigma$, the tail $\Sigma_\sigma \cap \{x \geq b\}$ is proper and complete in the direction $x \rightarrow \infty$: properness follows from $|X| \geq x$, and every curve escaping through $x \rightarrow \infty$ has infinite length. Completeness of the whole hypersurface is not established by these estimates. A possible continuation must solve the same geometric equation; changing parametrization does not remove the obstruction of Proposition 8.5. For $r = 1$, after the normalization $\kappa = 1/2$ of Kleene–Møller, the profile is defined on $[0, \infty)$, it is strictly convex there with $0 < u'_\sigma < \sigma$, and $u_\sigma(0) < \sqrt{2(n-1)}$ [21, Theorem 3(i),(iv), p. 3947]; in particular $x_\sigma < 0$. Moreover, the maximal curve containing the graph of u_σ is not embedded, so these ends are not parts of complete embedded self-shrinkers [21, Theorem 3(v) and Corollary 5, pp. 3947 and 3956].

9. CONCLUSIONS AND OPEN PROBLEMS

For every $1 \leq r \leq n-1$ and every positive conical slope we have constructed a smooth embedded rotational S_r -shrinking end. The strict negativity of the angular defect gives r -admissibility, the profile is unique under the sole assumption $u(x)/x \rightarrow \sigma > 0$, and its complete differentiable asymptotic expansion determines the geometry at infinity. Proposition 2.2 shows that for $r = n$ the only rotational graphical solution with a positive conical asymptote is the cone itself. On compact slope intervals the profiles depend real analytically on the slope, in weighted Hölder spaces controlling the full conical remainder, and they foliate distant exterior regions. Theorem A removes the symmetry assumption from the uniqueness statement under smooth multiplicity-one convergence to a circular cone, and Proposition 7.6 extends it to strictly admissible cones whose links have constant principal curvatures. These results identify the ends outside a compact set, but they do not classify their interior continuations. We conclude with several questions.

Intersections of the rotational family. Proposition 4.4 and Corollary 6.5 show that the profiles are ordered on distant exterior regions. Numerically, this ordering does not extend to all $x > 0$.

Remark 9.1. (Numerical.) For $\kappa = 1$, $(n, r) \in \{(3, 1), (3, 2), (4, 1), (5, 2), (5, 3), (6, 4)\}$, and consecutive slopes in $\{0.25, 0.5, 1, 2, 4\}$, the profiles u_σ and $u_{\sigma'}$ with $\sigma < \sigma'$ have exactly one detected intersection in $\{x > 0\}$. For instance, for $n = 3$ the intersection points are approximately $x = 0.47$ ($r = 1$) and $x = 0.47$ ($r = 2$) when $(\sigma, \sigma') = (0.5, 1)$, and $x = 0.19$ ($r = 1$) and $x = 0.17$ ($r = 2$) when $(\sigma, \sigma') = (1, 2)$. In all these cases $x_\sigma < 0$, and the computed values of $u_\sigma(0)$ decrease along the sampled slopes.

If one proves that $x_\sigma < 0$ and that $\sigma \mapsto u_\sigma(0)$ is strictly decreasing, then Proposition 4.4 and the intermediate value theorem show that u_σ and $u_{\sigma'}$ intersect in $(0, R(\sigma, \sigma'))$. In terms of the variation $J_\sigma = \partial_\sigma u_\sigma$, which satisfies the linearized equation

$$J''_\sigma = F_u(x, u_\sigma, u'_\sigma)J_\sigma + F_p(x, u_\sigma, u'_\sigma)J'_\sigma$$

and $J_\sigma = x + O(x^{-r})$ at infinity (Corollary 6.5), global ordering of the family is equivalent to $J_\sigma > 0$ on $(0, \infty)$ for every σ . Indeed, ordering gives $J_\sigma \geq 0$; at an interior zero x_0 of a nonnegative J_σ we would have $J'_\sigma(x_0) = 0$, and uniqueness for the linear equation would force $J_\sigma \equiv 0$, which is impossible. Conversely, integrating $J_\sigma > 0$ in σ gives ordering. More precisely, an intersection forces a zero of the variation: if $u_\sigma(x_0) = u_{\sigma'}(x_0)$ for some $x_0 > 0$ and $\sigma < \sigma'$, then

$$0 = \int_\sigma^{\sigma'} J_\tau(x_0) d\tau,$$

so $J_\tau(x_0) \leq 0$ for some $\tau \in (\sigma, \sigma')$; since $J_\tau(x) = x + O(x^{-r}) > 0$ for large x , the function J_τ has a zero in $(0, \infty)$. Thus each intersection observed in Remark 9.1 yields a zero of J_τ for some intermediate slope, and $J_\tau > 0$ fails for that τ .

Open question 9.2. Do the profiles u_σ and $u_{\sigma'}$, $\sigma \neq \sigma'$, always intersect in $\{x > 0\}$, and, if an intersection exists, is it unique?

Open question 9.3. For each $\sigma > 0$, does the variation $J_\sigma = \partial_\sigma u_\sigma$ have exactly one zero in $(0, \infty)$?

The two questions concern finite differences and infinitesimal variations, respectively. We do not assert their equivalence. Such an equivalence would require a separate comparison argument for the profile equation, including control of zeros approaching the boundary of the graphical interval.

The maximal graphical interval. The construction concerns the conical end, and Proposition 8.4 only describes the maximal graphical interval qualitatively.

Open question 9.4. Let $2 \leq r \leq n - 1$. Is $x_\sigma < 0$ for every $\sigma > 0$? When x_σ is finite, does u_σ approach a positive finite radius with $|u'_\sigma| \rightarrow \infty$, and what is the precise rate of curvature blow-up? At which points, if any, does the generating curve leave the closure of the r -admissible region? What other endpoint behaviors are possible, including approach to the axis or a second end? If a weak or singular continuation is considered, which notion of solution is appropriate and under what conditions does it exist?

If the endpoint has nonzero x_σ and positive finite radius with a vertical limiting tangent, a smooth continuation is ruled out by Proposition 8.5; reparametrization by arclength cannot regularize the meridian curvature. Endpoints with $x_\sigma = 0$ or vanishing radius require separate analysis and are not classified by that proposition.

For $r = 1$, the classical results of Kleene–Møller give $x_\sigma < 0$ and a nonembedded maximal generating curve [21, Theorem 3, p. 3947, and Corollary 5, p. 3956]; their proof uses geodesics of the Angenent metric. The obstruction at a positive-radius vertical tangent is different for $r \geq 2$, since the coefficient of the meridian curvature vanishes when $\lambda = 0$.

General cones.

Open question 9.5. For $2 \leq r \leq n - 1$, does uniqueness at infinity hold for smoothly asymptotically conical S_r -shrinking ends over every strictly r -admissible regular cone, without assuming that the principal curvatures of its link are constant?

Lemma 7.1 shows that the difference of two such ends vanishes to every algebraic order for an arbitrary regular cone. The remaining obstruction is therefore not asymptotic flatness but unique continuation at infinity for the effective operator. As explained in Remark 7.8, normalization of its radial principal coefficient generally produces an angularly dependent leading drift, which is not covered by the weakly conical almost-eigenfunction theory of Theorem 7.2. If the limiting cone lies on the boundary of the Gårding cone, one must in addition address the degeneration of the principal symbol.

FUNDING

The authors were partially supported by the Brazilian National Council for Scientific and Technological Development (CNPq), through Grant 303118/2022-9 to H.A., Grant 304381/2026-8 to M.B., and Grant 306189/2023-2 to G.S.N. All authors were also partially supported by the Coordination for the Improvement of Higher Education Personnel (CAPES), Brazil, Finance Code 001.

DATA AVAILABILITY STATEMENT

No data are associated with this manuscript. Figure 1 and Remarks 8.6 and 9.1 are based on numerical solutions of the terminal problems described in the caption of Figure 1; they are not certified and no proof relies on them.

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interest.

AI DISCLOSURE

A generative AI tool was used to assist with language editing, organization of the exposition, and discussion and checking of selected algebraic calculations. The authors take full responsibility for the mathematical statements, proofs, and references.

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