HALF-SPACE THEOREMS FOR TRANSLATING SOLITONS OF THE r-MEAN CURVATURE FLOW

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Abstract. In this paper, we establish nonexistence results for complete translating solitons of the r-mean curvature flow under suitable growth conditions on the $(r-1)$-mean curvature and on the norm of the second fundamental form. We first show that such solitons cannot be entirely contained in the complement of a right rotational cone whose axis of symmetry is aligned with the translation direction. We then relax the growth condition on the $(r-1)$-mean curvature and prove that properly immersed translating solitons cannot be confined to certain half-spaces opposite to the translation direction. We conclude the paper by showing that complete, properly immersed translating solitons satisfying appropriate growth conditions on the $(r-1)$-mean curvature cannot lie completely within the intersection of two transversal vertical half-spaces.

1. Introduction

Let $X_0 : \Sigma^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a n-dimensional Riemannian manifold Σ^n, with the second fundamental form

$$II(X,Y) = \langle A(X), Y\rangle N,$$

where $A : T\Sigma^n \to T\Sigma^n$ is its shape operator and N is a unit normal vector field. Letting $k_1, \ldots, k_n$ be the principal curvatures of the immersion, we define the r-mean curvatures as

$$\begin{cases} \sigma_0 = 1, \\ \sigma_r = \displaystyle\sum_{i_1<\cdots<i_r} k_{i_1}\cdots k_{i_r}, \quad \text{for} \quad 1 \le r \le n, \\ \sigma_r = 0, \ \text{for } r > n, \end{cases} \tag{1.1}$$

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where $(i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$. These functions appear naturally in the characteristic polynomial of A, since

$$\begin{aligned} \det(A - tI) &= \sigma_n - \sigma_{n-1}t + \sigma_{n-2}t^2 - \sigma_{n-3}t^3 + \cdots + (-1)^n t^n \\ &= \sum_{j=0}^{n} (-1)^j \sigma_{n-j} t^j. \end{aligned} \tag{1.2}$$

A family of isometric immersions $\mathcal{X} : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of the r-mean curvature flow if it satisfies the initial value problem

$$\begin{cases} \dfrac{\partial \mathcal{X}}{\partial t}(x, t) = \sigma_r(k_1(x, t), \ldots, k_n(x, t)) \cdot N(x, t), \\ \mathcal{X}(x, 0) = X_0(x). \end{cases} \tag{1.3}$$

Here, $k_1(x, t), \ldots, k_n(x, t)$ are the principal curvatures of the immersions $X_t := \mathcal{X}(\cdot, t)$ and $N(\cdot, t)$ are their normal vector fields. When $r = 1$, the r-mean curvature flow is called the mean curvature flow.

The r-mean curvature flow has been widely studied in recent decades, as we can see, for instance, in [5], [7], [8], [11], [15], [18], [20], [21], [26], [27], [32], [33], and [35].

Among the most significant solutions of (1.3) are the self-similar ones, in which the initial immersion X_0 evolves under the flow only through an isometry or a homothety in $\mathbb{R}^{n+1}$. Translating solitons, also called translators, form the subclass where the evolution consists solely of a translation in $\mathbb{R}^{n+1}$, i.e.,

$$\mathcal{X}(x, t) = X_0(x) + tV,$$

where $V \in \mathbb{R}^{n+1}$ is a unit vector, called the velocity vector of the translator. It can be easily proven that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a translating soliton of the r-mean curvature flow, then

$$\sigma_r(x) = \langle N(x), V \rangle. \tag{1.4}$$

A natural tool to study the r-mean curvature is the r-th Newton transformation $P_r : T\Sigma^n \to T\Sigma^n$, $0 \leq r \leq n-1$, defined recursively by

$$\begin{cases} P_0 = I, \\ P_r = \sigma_r I - AP_{r-1}, \quad r \geq 1, \end{cases} \tag{1.5}$$

where $I : T\Sigma^n \to T\Sigma^n$ is the identity operator. In the context of Differential Geometry, it first appeared in the work of Reilly [30] in the expressions of the variational integral formulas for functions $f(\sigma_0, \ldots, \sigma_n)$ of the elementary symmetric functions σ_i's. Since then, the Newton transformations have been widely used as a tool in the study of the r-mean curvature, as we can see, for example, in [1], [2], [3], [4], [6], [10], [13], [16], [23], [24], [31], and [34]. Since we are assuming that Σ^n has a global choice of N we have that P_r is globally defined.

Remark 1.1. Solving the recurrence, we can also write P_r as the polynomial

$$P_r = \sigma_r I - \sigma_{r-1} A + \sigma_{r-2} A^2 - \cdots + (-1)^r A^r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j. \tag{1.6}$$

Notice that the Newton transformation P_r is a "degree r version" of the characteristic polynomial (1.2) applied to A.

In this paper, we establish non-existence results for translating solitons of the r-mean curvature flow in certain regions of $\mathbb{R}^{n+1}$. We first prove a non-existence theorem for translating solitons of the r-mean curvature flow in the complement of the open cone

$$\mathcal{C}_{V,a} = \left\{ X \in \mathbb{R}^{n+1};\ \left\langle \frac{X}{\|X\|}, V \right\rangle > a,\ a \in (0,1) \right\}.$$

Theorem 1.1. *There are no complete, n-dimensional, translating solitons of the r-mean curvature flow $\Sigma^n \subset \mathbb{R}^{n+1}$ with velocity V with P_{r-1} positive semidefinite, contained in the complement of the open cone $\mathcal{C}_{V,a}$,*

$$(\mathcal{C}_{V,a})^c = \left\{ X \in \mathbb{R}^{n+1};\ \left\langle \frac{X}{\|X\|}, V \right\rangle \le a,\ a \in (0,1) \right\},$$

satisfying one of the following conditions:

(i) *Σ^n is properly immersed and*

$$\limsup_{\delta(x)\to\infty} \frac{\sigma_{r-1}(x)}{\delta(x)} < \frac{r(1-a)}{a(n-r+1)}, \tag{1.7}$$

where $\delta(x)$ denotes the extrinsic distance to a fixed point of $\mathbb{R}^{n+1}$;

(ii) *σ_{r-1} is bounded and*

$$\limsup_{\rho(x)\to\infty} \frac{\|A(x)\|}{\rho(x)\log(\rho(x))\log(\log(\rho(x)))} < \infty, \tag{1.8}$$

where $\rho(x)$ denotes the intrinsic distance to a fixed point of $\mathbb{R}^{n+1}$.

If $r = 1$, then $\sigma_{r-1} = \sigma_0 = 1$ and $P_{r-1} = P_0 = I$ (that is positive definite). Moreover, (1.7) is automatically satisfied. Therefore, we immediately obtain:

Corollary 1.1. *There are no complete, properly immersed, n-dimensional, translating solitons of the mean curvature flow $\Sigma^n \subset \mathbb{R}^{n+1}$, with velocity V, contained in the complement of the open cone $\mathcal{C}_{V,a}$,*

$$(\mathcal{C}_{V,a})^c = \left\{ X \in \mathbb{R}^{n+1}; \left\langle \frac{X}{\|X\|}, V \right\rangle \leq a,\ a \in (0,1) \right\}.$$

Remark 1.2. Clearly, condition (ii) in Theorem 1.1 can be used to replace the hypothesis of being properly immersed in Corollary 1.1 by the control of the second fundamental form given by (1.8), since, for $r = 1$, $\sigma_{r-1} = 1$.

Remark 1.3. Notice that the conditions given in Equations (1.7) and (1.8) are equivalent to the existence of positive constants $C, D > 0$ such that

$$\sigma_{r-1}(x) < \frac{r(1-a)}{a(n-r+1)}[\delta(x) + C]$$

and

$$\|A(x)\| \leq D\rho(x)\log(\rho(x))\log(\log(\rho(x))),$$

for $\delta(x) \gg 1$ and $\rho(x) \gg 1$ respectively.

If we restrict the region $(\mathcal{C}_{V,a})^c$ to a halfspace of the form

$$\mathcal{H}_W = \left\{ X \in \mathbb{R}^{n+1}; \langle X, W\rangle \leq 0,\ \langle V, W\rangle > 0,\ \|W\| = 1 \right\},$$

that is always contained in $(\mathcal{C}_{V,a})^c$ for any $a \in (0,1)$ such that $a \geq \langle V, W\rangle$, we can improve the hypothesis (1.7) in the case that Σ^n is properly immersed. This is the content of the next

Theorem 1.2. *There is no complete, n-dimensional, properly immersed, translating soliton of the r-mean curvature flow $\Sigma^n \subset \mathbb{R}^{n+1}$, with velocity V, P_{r-1} positive semidefinite, contained in the closed half-space*

$$\mathcal{H}_W = \left\{ X \in \mathbb{R}^{n+1}; \langle X, W\rangle \leq 0,\ \langle V, W\rangle > 0,\ \|W\| = 1 \right\},$$

and such that

$$\limsup_{\delta(x)\to\infty} \frac{\sigma_{r-1}(x)}{[\delta(x)]^2\log(\delta(x))\log(\log(\delta(x)))} < \infty, \tag{1.9}$$

where $\delta(x)$ denotes the extrinsic distance to a fixed point of $\mathbb{R}^{n+1}$.

Remark 1.4. Notice that the condition given in Equation (1.9) is equivalent to the existence of a positive constant $D > 0$ such that

$$\sigma_{r-1}(x) \leq D[\delta(x)]^2\log(\delta(x))\log(\log(\delta(x)))$$

for $\delta(x) \gg 1$. This growth condition for $\sigma_{r-1}(x)$ is clearly better than that given by Equation (1.7) in Theorem 1.1.

If $r = 1$, then $\sigma_{r-1} = \sigma_0 = 1$ and (1.9) is automatically satisfied. Thus, we obtain the following

Corollary 1.2. *There are no complete, properly immersed, n-dimensional, translating solitons of the mean curvature flow $\Sigma^n \subset \mathbb{R}^{n+1}$, with velocity V, in a closed half-space*

$$\mathcal{H}_W = \left\{X \in \mathbb{R}^{n+1}; \langle X, W\rangle \leq 0,\ \langle V, W\rangle > 0,\ \|W\| = 1\right\}.$$

Remark 1.5. Corollaries 1.2 and 1.1 are stated in [25] without the hypothesis that Σ^n is properly immersed. Unfortunately, there was an error in their proof. In order to apply the Omori–Yau maximum principle, the authors claimed that translating solitons of the mean curvature flow have Ricci curvature bounded below by $-1/4$ (Equation (3.1), p. 5), but, in their argument, the sign in the Gauss equation is reversed, leading to an incorrect estimate.

Example 1.1. For $r = 1$, the Grim Reaper cylinder, the bowl soliton, and the translating catenoids are examples of properly immersed translating solitons that are not contained in $\mathcal{H}_W$ or $(\mathcal{C}_{V,a})^c$ for $V = E_{n+1}$. Moreover, this property persists under any translation in the direction of V, i.e., for $\Sigma^n + tV$, $t \in \mathbb{R}$. For the Grim Reaper cylinder, this follows from the fact that the curve $y = -\log(\cos x)$ has two vertical asymptotes, implying that its graph intersects every rotational cone with vertex at the origin and axis E_{n+1}. By the same reason, no vertical translation in the E_{n+1}-direction can place it entirely within any $\mathcal{H}_W$. On the other hand, one can always find a suitable translation such that these hypersurfaces lie in the complements $\mathbb{R}^{n+1} \setminus \mathcal{H}_W$ and $\mathcal{C}_{V,a}$ (see Figure 1). The same phenomenon occurs for the bowl soliton and the translating catenoids, whose asymptotic expansion as $R(x)$ approaches infinity is

$$\frac{R(x)^2}{2(n-1)} - \log(R(x)) + O(R(x)^{-1}),$$

where $R(x)$ denotes the Euclidean distance from $x \in \mathbb{R}^n$ to the origin (remember these hypersurfaces are radial graphs over a subset of $\mathbb{R}^n$).

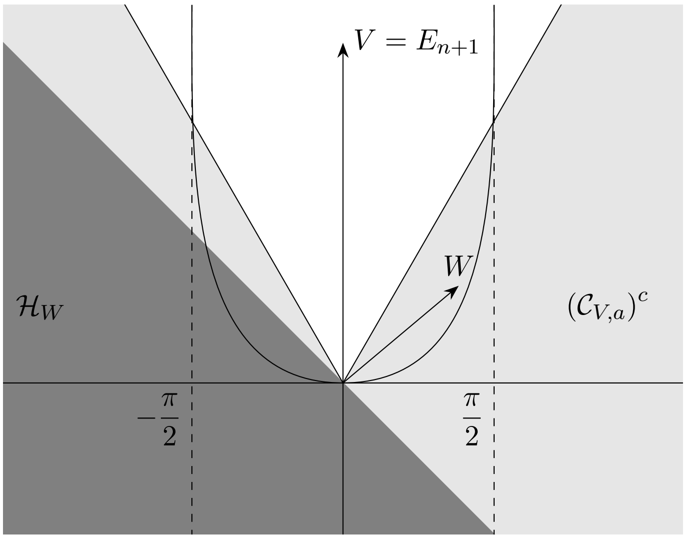

FIGURE 1. The Grim Reaper cylinder and the sets $\mathcal{H}_V$ and $(\mathcal{C}_{V,a})^c$.

Example 1.2. If $r > 1$, the r-bowl soliton and the r-translating catenoids classified by R. de Lima and G. Pipoli in [28] share the same properties as their counterparts for $r = 1$. Indeed, they are properly embedded (despite, for $r > 1$, the r-translating catenoids are not complete), and neither they nor any of their translations in the direction of $V = E_{n+1}$ are contained in any $\mathcal{H}_W$ or $(\mathcal{C}_{V,a})^c$. Moreover, one can always find a suitable translation such that these hypersurfaces lie in the complements $\mathbb{R}^{n+1} \setminus \mathcal{H}_W$ and $\mathcal{C}_{V,a}$. In fact, de Lima and Pipoli proved that the angle function $\Theta = \langle N, E_{n+1} \rangle$ converges to 1 as the Euclidean distance $R(x)$ from $x \in \mathbb{R}^n$ to the origin tends to infinity. It follows that these hypersurfaces intersect every rotational cone with fixed angle, and thus cannot be contained in any $(\mathcal{C}_{V,a})^c$ for $V = E_{n+1}$.

In order to introduce our next result, we need the following

Definition 1.1. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a translating soliton of the r-mean curvature flow with velocity vector V, this is, $\sigma_r = \langle N, V \rangle$, where N is the normal vector field of Σ^n.

(i) Given $B, W \in \mathbb{R}^{n+1}$, we say that a halfspace

$$\mathcal{H} := \mathcal{H}_{(B,W)} := \{X \in \mathbb{R}^{n+1}; \langle X - B, W \rangle \geq 0\},$$

is *vertical* if $W \perp V$;

(ii) Two halfspaces

$$\mathcal{H}_i := \mathcal{H}_{(B_i,W_i)} := \{X \in \mathbb{R}^{n+1}; \langle X - B_i, W_i\rangle \geq 0\}, \quad i = 1, 2,$$

are *transversal* if W_1 and W_2 are linearly independent.

We conclude this paper proving a nonexistence result for translating solitons of the r-mean curvature flow in intersection of two vertical half-spaces. This result is a generalization of the "bi-halfspace" Theorem 1.1 of [17] by Chini and Møller.

Theorem 1.3. *There is no complete, properly immersed, n-dimensional, translating soliton of the r-mean curvature flow $\Sigma^n \subset \mathbb{R}^{n+1}$, contained in the intersection of two transversal vertical half-spaces, such that $P_{r-1} \geq \varepsilon I$, for some $\varepsilon > 0$, and*

$$\limsup_{\delta(x)\to\infty} \frac{\sigma_{r-1}(x)}{[\delta(x)]^2 \log(\delta(x)) \log(\log(\delta(x)))} < \infty, \tag{1.10}$$

where $\delta : \Sigma^n \to \mathbb{R}_+$ is the extrinsic distance to a fixed point.

Remark 1.6. We say that $P_{r-1} \geq \varepsilon I$, for some $\varepsilon > 0$, if $\langle P_{r-1}(v), v\rangle \geq \varepsilon \|v\|^2$, for any $v \in T\Sigma^n$.

In Theorem 1.3, if we take $r = 1$, then $\sigma_0 = 1$ and $P_0 = I$. Moreover, (1.10) is automatically satisfied. Therefore, we immediately obtain:

Corollary 1.3 (Chini and Møller [17])**.** *There does not exist any complete, properly immersed, n-dimensional translating soliton of the mean curvature flow, $\Sigma^n \subset \mathbb{R}^{n+1}$ that is contained in the intersection of two transversal vertical half-spaces.*

Remark 1.7. Clearly, the bowl soliton, the translating catenoids, and their counterparts for $r > 1$, classified in [28], intersect every vertical hyperplane and therefore cannot be contained in the intersection of two transversal vertical half-spaces. On the other hand, when $r = 1$, the Grim Reaper cylinder lies between two parallel hyperplanes; however, once one of these hyperplanes is fixed, the cylinder intersects every other vertical hyperplane transversal to it. This shows that the Grim Reaper cylinder is not contained in the intersection of two transversal vertical half-spaces either.

2. Omori-Yau type maximum principles

The celebrated Omori-Yau maximum principle can be considered in a variety of differential operators acting on smooth functions of a Riemannian manifold Σ^n other than the Laplacian. In the following, we use the maximum principle found in [12] which we include a complete proof here (with more details) for the sake of completeness.

Let Σ^n be a n-dimensional Riemannian manifold, $f : \Sigma^n \to \mathbb{R}$ be a class $\mathcal{C}^2$ function, and $\phi : T\Sigma^n \to T\Sigma^n$ be a linear symmetric tensor. Define the second-order differential operator

$$\mathcal{L}_\phi f := \operatorname{trace}(\phi \circ \operatorname{hess} f) - \langle Z, \nabla f\rangle,$$

where Z is a vector field defined on Σ^n with $\sup_{\Sigma^n} \|Z\| < \infty$. Here, $\operatorname{hess} f : T\Sigma^n \to T\Sigma^n$ is the linear operator $\operatorname{hess} f(W) = \nabla_W \nabla f$, associated to the hessian quadratic form $\operatorname{Hess} f$, i.e., $\operatorname{Hess} f(W_1, W_2) = \langle \operatorname{hess} f(W_1), W_2\rangle$.

Lemma 2.1 (G. P. Bessa and L. Pessoa, [12]). *Let Σ^n be an n-dimensional complete Riemannian manifold, $\phi : T\Sigma^n \to T\Sigma^n$ be a symmetric and positive semidefinite linear tensor, and Z be a bounded vector field on Σ^n. If there exists a positive function $\gamma \in \mathcal{C}^2(\Sigma^n)$ and $G : [0,\infty) \to [0,\infty)$ such that*

(i) $G(0) > 0$, $G'(t) \geq 0$, $G(t)^{-1/2} \notin L^1([0,\infty))$;

(ii) $\gamma(x) \to \infty$ *when* $x \to \infty$;

(iii) $\exists A > 0$ *such that* $\|\nabla\gamma\| \leq A\sqrt{G(\gamma)}\left(\int_a^\gamma \frac{ds}{\sqrt{G(s)}} + 1\right)$ *off a compact set, for some* $a \gg 1$ *;*

(iv) $\exists B > 0$ *such that* $\operatorname{trace}(\phi \circ \operatorname{hess}\gamma) \leq B\sqrt{G(\gamma)}\left(\int_a^\gamma \frac{ds}{\sqrt{G(s)}} + 1\right)$ *off a compact set, for some* $a \gg 1$*;*

then, for every function $u \in \mathcal{C}^2(\Sigma^n)$ satisfying

$$\lim_{x\to\infty} \frac{u(x)}{\varphi(\gamma(x))} = 0, \quad \textit{for} \quad \varphi(t) = \ln\left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1\right),\ t \in [0,\infty), \tag{2.1}$$

there exists a sequence of points $\{x_k\}_k \subset \Sigma^n$ such that

$$\|\nabla u(x_k)\| < \frac{1}{k} \quad \textit{and} \quad \mathcal{L}_\phi u(x_k) < \frac{1}{k}. \tag{2.2}$$

Moreover, if instead of (2.1) *we assume that* $u^* = \sup_{\Sigma^n} u < \infty$*, then*

$$\lim_{k\to\infty} u(x_k) = u^*.$$

Proof. Let

$$f_k(x) = u(x) - \varepsilon_k \varphi(\gamma(x)),$$

for each positive integer k, where $\varepsilon_k > 0$ is a sequence satisfying $\varepsilon_k \to 0$, when $k \to \infty$. Since, for a fixed $x_0 \in \Sigma^n$, the sequence $\{f_k(x_0)\}_k$, defined by $f_k(x_0) = u(x_0) - \varepsilon_k \varphi(\gamma(x_0))$ is bounded, adding a positive constant to the function u, if necessary, we may assume that $f_k(x_0) > 0$ for every $k > 0$. Notice that, by (2.1),

$$\lim_{x\to\infty} \frac{f_k(x)}{\varphi(\gamma(x))} = \lim_{x\to\infty} \frac{u(x)}{\varphi(\gamma(x))} - \varepsilon_k = -\varepsilon_k < 0,$$

which implies that f_k is non-positive out of a compact set $\Omega_k \subset \Sigma^n$ containing x_0. Thus, f_k achieves its maximum at a point $x_k \in \Omega_k$ for each $k \geq 1$. Now, notice that

$$\varphi'(t) = \left[\sqrt{G(t)}\left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1\right)\right]^{-1} > 0 \tag{2.3}$$

and

$$\begin{aligned} \varphi''(t) = &- \left[\frac{G'(t)}{2\sqrt{G(t)}}\left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1\right) + 1\right] \times \\ &\times \left[\sqrt{G(t)}\left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1\right)\right]^{-2} < 0. \end{aligned} \tag{2.4}$$

Since

$$\nabla f_k = \nabla u - \varepsilon_k \varphi'(\gamma)\nabla\gamma$$

and

$$\operatorname{Hess} f_k(W,W) = \operatorname{Hess} u(W,W) - \varepsilon_k \left[\varphi'(\gamma) \operatorname{Hess}\gamma(W,W) + \varphi''(\gamma)\langle W, \nabla\gamma\rangle^2\right],$$

we have, at x_k, that

$$\nabla u(x_k) = \varepsilon_k \varphi'(\gamma(x_k))\nabla\gamma(x_k) \tag{2.5}$$

and

$$\begin{aligned} \operatorname{Hess} u(x_k)(W,W) &\leq \varepsilon_k \varphi'(\gamma(x_k)) \operatorname{Hess}\gamma(x_k)(W,W) \\ &\quad + \varepsilon_k \varphi''(\gamma(x_k))\langle W, \nabla\gamma(x_k)\rangle^2 \\ &\leq \varepsilon_k \varphi'(\gamma(x_k)) \operatorname{Hess}\gamma(x_k)(W,W). \end{aligned} \tag{2.6}$$

Now, Equation (2.5) and hypothesis (iii) imply

$$\|\nabla u(x_k)\| = \varepsilon_k \varphi'(x_k)\|\nabla\gamma(x_k)\| \leq \varepsilon_k A < \frac{1}{k},$$

for $\varepsilon_k < \frac{1}{kA}$. On the other hand, letting $\{e_1, \ldots, e_n\}$ be an orthonormal frame formed with eigenvectors of $\phi : T\Sigma^n \to T\Sigma^n$, with nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$, we have, using (2.3) and hypothesis (iii) and (iv),

$$\begin{aligned}
\mathcal{L}_\phi u(x_k) &= \sum_{i=1}^n \langle \operatorname{hess} u(x_k)(e_i), \phi(e_i)\rangle - \langle Z(x_k), \nabla u(x_k)\rangle \\
&= \sum_{i=1}^n \lambda_i \langle \operatorname{hess} u(x_k)(e_i), e_i\rangle - \langle Z(x_k), \nabla u(x_k)\rangle \\
&= \sum_{i=1}^n \lambda_i \operatorname{Hess} u(x_k)(e_i, e_i) - \langle Z(x_k), \nabla u(x_k)\rangle \\
&\leq \varepsilon_k \varphi'(\gamma(x_k)) \sum_{i=1}^n \lambda_i \operatorname{Hess}\gamma(x_k)(e_i, e_i) \\
&\quad - \varepsilon_k \varphi'(\gamma(x_k))\langle Z(x_k), \nabla\gamma(x_k)\rangle \\
&= \varepsilon_k \varphi'(\gamma(x_k)) \operatorname{trace}(\phi \circ \operatorname{hess}\gamma)(x_k) \\
&\quad - \varepsilon_k \varphi'(\gamma(x_k))\langle Z(x_k), \nabla\gamma(x_k)\rangle \\
&\leq \varepsilon_k \left(B + A \sup_{\Sigma^n} \|Z\|\right) < \frac{1}{k},
\end{aligned}$$

if we take

$$\varepsilon_k < \frac{1}{k \max\{A, B + A\sup_{\Sigma^n} \|Z\|\}}.$$

If $= u^* = \sup_{\Sigma^n} u(x) < \infty$, then, given an arbitrary integer $m > 0$, let $y_m \in \Sigma^n$ such that

$$u(y_m) > u^* - \frac{1}{2m}.$$

This gives

$$\begin{aligned}
f_k(x_k) = u(x_k) - \varepsilon_k\gamma(x_k) &\geq f_k(y_m) \\
&= u(y_m) - \varepsilon_k\gamma(y_m) \\
&> u^* - \frac{1}{2m} - \varepsilon_k\gamma(y_m),
\end{aligned}$$

which implies

$$\begin{aligned}
u(x_k) &> u^* - \frac{1}{2m} - \varepsilon_k\gamma(y_m) + \varepsilon_k\gamma(x_k) \\
&> u^* - \frac{1}{2m} - \varepsilon_k\gamma(y_m).
\end{aligned}$$

Now, choosing k_m such that $\varepsilon_{k_m}\gamma(y_m) < \frac{1}{2m}$, we obtain that

$$u(x_{k_m}) > u^* - \frac{1}{m}.$$

Thus, by replacing x_k by x_{k_m} if necessary, we can conclude that

$$\lim_{k\to\infty} u(x_k) = u^*.$$

□

Remark 2.1. The typical examples of functions G satisfying condition (i) of Lemma 2.1 are given by

$$G(t) = t^2 \prod_{j=1}^{N} \left(\log^{(j)}(t)\right)^2, \quad t \gg 1, \tag{2.7}$$

where $\log^{(j)}(t)$ denotes the j-th iterate of $\log t$.

In the following, we apply Lemma 2.1 to the context of isometric immersions. Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be an isometric immersion and P_r the r-th Newton transformation defined by (1.5), p.2. We introduce the functional operator

$$L_{r-1}f = \mathrm{trace}(P_{r-1} \circ \mathrm{hess}\, f), \quad f \in \mathcal{C}^2(\Sigma^n).$$

This operator is important in the study of σ_r as a generalization of the Laplacian, see, for example [1], [2], [3], [4], [6], [10], [13], [16], [23], [24], [31], and [34].

Applying Lemma 2.1 to the operator L_{r-1}, we obtain

Theorem 2.1. *Let Σ^n be a complete hypersurface of $\mathbb{R}^{n+1}$ and $A : T\Sigma^n \to T\Sigma^n$ be its shape operator. Assume that the $(r-1)$-th Newton transformation P_{r-1} is positive semidefinite. If one of the following conditions holds:*

(i) *σ_{r-1} is bounded and*

$$\limsup_{\rho(x)\to\infty} \frac{\|A(x)\|}{\rho(x)\log(\rho(x))\log(\log(\rho(x)))} < \infty, \tag{2.8}$$

where $\rho : \Sigma^n \to \mathbb{R}_+$ is the intrinsic distance on Σ^n to a fixed point;

(ii) *or Σ^n is properly immersed,*

$$\limsup_{\delta(x)\to\infty} \frac{|\sigma_{r-1}(x)|}{[\delta(x)]^2\log(\delta(x))\log(\log(\delta(x)))} < \infty, \tag{2.9}$$

and

$$\limsup_{\delta(x)\to\infty} \frac{|\sigma_r(x)|}{\delta(x)\log(\delta(x))\log(\log(\delta(x)))} < \infty, \tag{2.10}$$

where $\delta : \Sigma^n \to \mathbb{R}_+$ is the extrinsic distance of $\mathbb{R}^{n+1}$ to a fixed point, restricted to Σ^n,

then, for any class $\mathcal{C}^2$ function $f : \Sigma^n \to \mathbb{R}$, bounded from above, there exists a sequence of points $\{x_k\}_k \subset \Sigma^n$ such that

$$\begin{cases} \lim_{k\to\infty} f(x_k) = \sup_{\Sigma^n} f, \\ \|\nabla f(x_k)\| \leq \dfrac{1}{k}, \\ L_{r-1} f(x_k) \leq \dfrac{1}{k}. \end{cases} \tag{2.11}$$

Proof. To prove the first part, under the hypothesis (i), let us follow Example 1.13 of [29]. Let $\gamma(x) = \rho(x)^2 = [\mathrm{dist}_\Sigma(x,p_0)]^2$, where $\mathrm{dist}_\Sigma(x,p_0)$ is the intrinsic distance of Σ^n to a point $p_0 \in \Sigma^n$. Then γ is smooth in $\Sigma^n \setminus (\{p_0\} \cup \mathrm{cut}(p_0))$, where $\mathrm{cut}(p_0)$ denotes the cut locus of p_0. Since, for the points at the cut locus, we can apply the Calabi's trick, we will work only with the points where γ is smooth.

If $\|A(x)\|^2 \leq CG_0(\rho(x))$, for some smooth function $G_0 : [0,+\infty) \to \mathbb{R}$, even at the origin, then, by the Gauss equation, the sectional curvatures K_Σ of Σ^n satisfies

$$K_\Sigma \geq -2\|A\|^2 \geq -2CG_0(\rho).$$

This implies, following the proof of Example 1.13 of [29] step-by step, that

$$\mathrm{Hess}\,\gamma(Y,Y) \leq B\gamma^{1/2} G_0(\gamma^{1/2})^{1/2}\|Y\|^2$$

for some $B > 0$ and $\rho(x)$ sufficiently large. If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of P_{r-1} with eigenvetors $e_1,\ldots,e_n$, we have

$$\begin{aligned} L_{r-1}\gamma &= \mathrm{trace}(P_{r-1} \circ \mathrm{hess}\,\gamma) \\ &= \sum_{j=1}^n \langle P_{r-1}(\mathrm{hess}\,\gamma(e_j)), e_j\rangle \\ &= \sum_{j=1}^n \langle \mathrm{hess}\,\gamma(e_j), P_{r-1}(e_j)\rangle \\ &= \sum_{j=1}^n \lambda_j \langle \mathrm{hess}\,\gamma(e_j), (e_j)\rangle \\ &= \sum_{j=1}^n \lambda_j \,\mathrm{Hess}\,\gamma(e_j,e_j) \\ &\leq B(\mathrm{trace}\,P_{r-1})\gamma^{1/2} G_0(\gamma^{1/2})^{1/2}. \end{aligned} \tag{2.12}$$

Now, taking $G(t) = (t\log t)^2$ and $\gamma = \rho^2$, in order to apply Lemma 2.1, observe that

$$
\begin{aligned}
P(\gamma) &:= \sqrt{G(\gamma)}\left(\int_a^\gamma \frac{ds}{\sqrt{G(s)}} + 1\right) \\
&= \gamma\log(\gamma)[\log(\log(\gamma)) - \log(\log(a)) + 1] \\
&= 2\rho^2\log\rho[\log(\log(2\rho) - \log(\log(a)) + 1] \\
&= 2\rho^2\log\rho[\log(\log(\rho) + \log 2 - \log(\log(a)) + 1] \\
&= 2\rho^2\log\rho\log(\log\rho),
\end{aligned}
\tag{2.13}
$$

for $a = e^{2e}$. Clearly $\|\nabla\gamma\| = 2\rho \le P(\gamma)$ for ρ sufficiently large. On the other hand, the condition (iv) of Lemma 2.1 holds if we assume that σ_{r-1} is bounded and take

$$
G_0(t) = (t\log t\log(\log t))^2. \tag{2.14}
$$

Indeed, since $\operatorname{trace}(P_{r-1}) = (n-r+1)\sigma_{r-1}$ (see Lemma 2.1, p.279 of [9]), Equation (2.12) gives

$$
L_{r-1}\gamma \le B_1\sigma_{r-1}\rho G_0(\rho)^{1/2} \le B_2\rho^2\log\rho\log(\log\rho).
$$

In order to prove the second part, i.e., under the hypothesis (ii), we will follow the ideas of Example 1.14 of [29]. Indeed, assume that Σ^n is properly immersed and let $\gamma(x) = [\delta(x)]^2 = \|X(x) - X(p_0)\|^2$. By using (2.13) with δ in the place of ρ, we have $\|\nabla\gamma\| \le 2\delta \le P(\gamma)$ for large values of δ. On the other hand,

$$
\begin{aligned}
\operatorname{Hess}\gamma(Y,Z) &= \overline{\operatorname{Hess}}\,\delta^2(Y,Z) + \langle A(X), Y\rangle\langle N, \overline{\nabla}\delta^2\rangle \\
&= 2\langle X, Y\rangle + \langle A(X), Y\rangle\langle N, \overline{\nabla}\delta^2\rangle,
\end{aligned}
\tag{2.15}
$$

where $\overline{\nabla}$ and $\overline{\operatorname{Hess}}$ denote the gradient and the hessian of $\mathbb{R}^{n+1}$, respectively. This implies

$$
\begin{aligned}
L_{r-1}\gamma &= 2(\operatorname{trace}P_{r-1}) + 2(\operatorname{trace}(A\circ P_{r-1}))\langle N, \delta\overline{\nabla}\delta\rangle, \\
&= 2(n-r+1)\sigma_{r-1} + 2r\sigma_r\langle N, \delta\overline{\nabla}\delta\rangle,
\end{aligned}
\tag{2.16}
$$

where, in the last equality, we used that

$$
\operatorname{trace}(P_{r-1}) = (n-r+1)\sigma_{r-1} \quad \text{and} \quad \operatorname{trace}(A\circ P_{r-1}) = r\sigma_r, \tag{2.17}
$$

(see Lemma 2.1, p.279 of [9]). If

$$
|\sigma_{r-1}| \le B_1\gamma^{1/2}G_0(\gamma^{1/2})^{1/2} \quad \text{and} \quad |\sigma_r| \le B_2G_0(\gamma^{1/2})^{1/2}
$$

for large $\delta(x)$, and for $G_0(t)$ given by (2.14), then

$$\begin{aligned}|L_{r-1}\gamma| &\leq 2(n-r+1)|\sigma_{r-1}| + 2r|\sigma_r|\gamma^{1/2}\\ &\leq 2(n-r+1)B_1\gamma^{1/2}G_0(\gamma^{1/2})^{1/2} + 2rB_2\gamma^{1/2}G_0(\gamma^{1/2})^{1/2}\\ &=: B_3\gamma^{1/2}G_0(\gamma^{1/2})^{1/2}\\ &\leq B_4P(\gamma).\end{aligned}$$

The result then follows by applying Lemma 2.1 for $G(t) = (t\log t)^2$. □

Remark 2.2. Clearly, Theorem 2.1 holds replacing, $G(t) = (t\log t)^2$ by those given in Equation (2.7) of Remark 2.1. Our choice was aesthetic.

3. proof of Theorem 1.1

Proof of Theorem 1.1. Let us denote the extrinsic distance to the origin by $\delta_0(x) = \|X(x)\|$. From $[\delta_0(x)]^2 = \langle X(x), X(x)\rangle$ we have

$$\delta_0\nabla\delta_0 = X^\top \quad \text{and} \quad \|\nabla\delta_0\| = \frac{\|X^\top\|}{\|X\|} \leq 1. \tag{3.1}$$

On the other hand, by (2.16) and by the definition of L_{r-1},

$$\frac{1}{2}L_{r-1}\delta_0^2 = (n-r-1)\sigma_{r-1} + r\sigma_r\langle X, N\rangle \tag{3.2}$$

and

$$\frac{1}{2}L_{r-1}\delta_0^2 = \delta_0 L_r\delta_0 + \langle P_{r-1}(\nabla\delta_0), \nabla\delta_0\rangle. \tag{3.3}$$

Combining Equations (3.1), (3.2), and (3.3) gives

$$L_{r-1}\delta_0 = \frac{1}{\delta_0}[(n-r+1)\sigma_{r-1} + r\sigma_r\langle X, N\rangle] - \frac{1}{\delta_0^3}\langle P_{r-1}(X^\top), X^\top\rangle. \tag{3.4}$$

Suppose by contradiction that a translating soliton $\Sigma^n \subset \mathbb{R}^{n+1}$ satisfying the hypotheses of Theorem 1.1 is contained in the complement of a cone $\mathcal{C}_{V,a}$, this is

$$\Sigma^n \subset (\mathcal{C}_{V,a})^c = \left\{X \in \mathbb{R}^{n+1};\ \left\langle \frac{X}{\|X\|}, V\right\rangle \leq a,\ a \in (0,1)\right\}.$$

Define the function $\psi : \Sigma^n \to \mathbb{R}$ by

$$\psi(x) := \langle X(x), V\rangle - a\|X(x)\| \leq 0. \tag{3.5}$$

Since for any $U_1, U_2 \in T\Sigma^n$,

$$U_1(\langle X, V\rangle) = \langle U_1, V\rangle \quad \text{and} \quad U_2U_1(\langle X, V\rangle) = \langle \overline{\nabla}_{U_2}U_1, V\rangle, \tag{3.6}$$

we have $\nabla\langle X, V\rangle = V^\top$. This gives

$$\nabla\psi = V^\top - a\nabla\delta_0,$$

and then, by (3.1),

$$\|\nabla\psi\| = \|V^\top - a\nabla\delta_0\| \geq \left| \|V^\top\| - a\frac{\|X^\top\|}{\|X\|}\right|.$$

Now, choosing an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$, defined on Σ^n, formed with the eigenvectors of P_{r-1} corresponding to eigenvalues λ_i, we have that

$$\begin{aligned}
\mathrm{Hess}\langle X, V\rangle(e_i, e_j) &= e_i e_j(\langle X, V\rangle) - \nabla_{e_i} e_j(\langle X, V\rangle) \\
&= \langle \overline{\nabla}_{e_i} e_j, V\rangle - \langle \nabla_{e_i} e_j, V\rangle \\
&= \langle B(e_i, e_j), V\rangle \\
&= \langle A(e_i), e_j\rangle\langle N, V\rangle.
\end{aligned}$$

Recalling that (see Lemma 2.1, p.279 of [9])

$$\mathrm{trace}(P_{r-1}) = (n-r+1)\sigma_{r-1} \quad \text{and} \quad \mathrm{trace}(A\circ P_{r-1}) = r\sigma_r, \tag{3.7}$$

we have

$$\begin{aligned}
L_{r-1}\langle X, V\rangle &= \mathrm{trace}(P_{r-1}\circ \mathrm{hess}\langle X, V\rangle) \\
&= \sum_{i=1}^n \langle P_{r-1}(\mathrm{hess}\langle X, V\rangle(e_i)), e_i\rangle \\
&= \sum_{i=1}^n \langle \mathrm{hess}\langle X, V\rangle(e_i), P_{r-1}(e_i)\rangle \\
&= \sum_{i=1}^n \lambda_i\langle \mathrm{hess}\langle X, V\rangle(e_i), (e_i)\rangle \\
&= \sum_{i=1}^n \lambda_i \,\mathrm{Hess}\langle X, V\rangle(e_i, e_i) \\
&= \sum_{i=1}^n \lambda_i\langle A(e_i), e_i\rangle\langle N, V\rangle \\
&= \sum_{i=1}^n \langle A(e_i), P_{r-1}(e_i)\rangle\langle N, V\rangle \\
&= \sum_{i=1}^n \langle e_i, AP_{r-1}(e_i)\rangle\langle N, V\rangle \\
&= \mathrm{trace}(A\circ P_{r-1})\langle N, V\rangle \\
&= r\sigma_r\langle N, V\rangle.
\end{aligned} \tag{3.8}$$

This implies,

$$\begin{aligned} L_{r-1}\psi &= L_{r-1}\langle X, V\rangle - aL_{r-1}\delta_0 \\ &= r\sigma_r\langle N, V\rangle - aL_{r-1}\delta_0 \\ &= r\sigma_r^2 - aL_{r-1}\delta_0. \end{aligned}$$

Since ψ is bounded from above, and assuming either condition (i) or (ii) in the statement of Theorem 1.1, we may apply Theorem 2.1, p.11. Indeed, if (1.7) or (1.8) hold, then (2.8) or (2.9) follows, respectively, while (2.10) is automatically satisfied since $|\sigma_r| \leq 1$ for translating solitons. Therefore, there exists a sequence of points $\{x_k\}_k \subset \Sigma^n$ such that

$$\lim_{k\to\infty} \psi(x_k) = \sup_{\Sigma^n} \psi \leq 0, \tag{3.9}$$

$$\frac{1}{k} > \|\nabla\psi(x_k)\| \geq \left| \|V^\top(x_k)\| - a\frac{\|x_k^\top\|}{\|x_k\|} \right|, \tag{3.10}$$

and

$$\frac{1}{k} > L_{r-1}\psi(x_k) = r[\sigma_r(x_k)]^2 - aL_{r-1}\delta_0(x_k). \tag{3.11}$$

By (3.10), it holds

$$\|V^\top(x_k)\| - \frac{1}{k} \leq a\frac{\|x_k^\top\|}{\|x_k\|} < \frac{1}{k} + \|V^\top(x_k)\|. \tag{3.12}$$

If $\sigma_r(x_k) = \langle N(x_k), V\rangle \to 0$, when $k \to \infty$, then $\|V^\top(x_k)\| \to 1$. Thus, by (3.12),

$$\lim_{k\to\infty} \frac{\|x_k^\top\|}{\|x_k\|} = \frac{1}{a} > 1,$$

which is an absurd.

In the general case, since $|\sigma_r| = |\langle N, V\rangle| \leq 1$, there exists a subsequence $\{x_{k_l}\}$ such that $\sigma_r(x_{k_l})$ converges for $l \to \infty$. This implies that

$$\|V^\top(x_{k_l})\|^2 = 1 - \langle V, N(x_{k_l})\rangle^2 = 1 - [\sigma_r(x_{k_l})]^2$$

converges, and by (3.12), the sequence $\|x_{k_l}^\top\|/\|x_{k_l}\|$ also converges. Therefore,

$$\begin{aligned}\lim_{l\to\infty}\left(1-[\sigma_r(x_{k_l})]^2\right) &= \lim_{l\to\infty}\left(1-\langle V, N(x_{k_l})\rangle^2\right)\\ &= \lim_{l\to\infty}\|V^\top(x_{k_l})\|^2\\ &= a^2\lim_{l\to\infty}\frac{\|x_{k_l}^\top\|^2}{\|x_{k_l}\|^2}\\ &= a^2\lim_{l\to\infty}\left(1-\frac{\langle x_{k_l}, N(x_{k_l})\rangle^2}{\|x_{k_l}\|^2}\right).\end{aligned}$$

This implies

$$\lim_{l\to\infty}[\sigma_r(x_{k_l})]^2\left(1-[\sigma_r(x_{k_l})]^2\right) = a^2\lim_{l\to\infty}[\sigma_r(x_{k_l})]^2\left(1-\frac{\langle x_{k_l}, N(x_{k_l})\rangle^2}{\|x_{k_l}\|^2}\right),$$

i.e.,

$$\begin{aligned}&\lim_{l\to\infty}\left([\sigma_r(x_{k_l})]^4-(1-a^2)[\sigma_r(x_{k_l})]^2\right)\\ &\qquad = a^2\lim_{l\to\infty}[\sigma_r(x_{k_l})]^2\left\langle\frac{x_{k_l}}{\|x_{k_l}\|}, N(x_{k_l})\right\rangle^2.\end{aligned}\tag{3.13}$$

In particular,

$$\lim_{l\to\infty}\left([\sigma_r(x_{k_l})]^2-(1-a^2)\right)\geq 0.$$

On the other hand, by (3.11) and (3.4), we have

$$\begin{aligned}\frac{1}{k} &> L_{r-1}\psi(x_k)\\ &= r[\sigma_r(x_k)]^2 - aL_{r-1}\|x_k\|\\ &= r[\sigma_r(x_k)]^2-\frac{a}{\|x_k\|}[(n-r+1)\sigma_{r-1}(x_k)+r\sigma_r(x_k)\langle x_k, N(x_k)\rangle]\\ &\quad+\frac{1}{\|x_k\|^3}\langle P_{r-1}x_k^\top, x_k^\top\rangle\\ &\geq r[\sigma_r(x_k)]^2-\frac{a}{\|x_k\|}[(n-r+1)\sigma_{r-1}(x_k)+r\sigma_r(x_k)\langle x_k, N(x_k)\rangle]\\ &\geq r[\sigma_r(x_k)]^2-a(n-r+1)\frac{\sigma_{r-1}(x_k)}{\|x_k\|}-ar\left|\sigma_r(x_k)\left\langle\frac{x_k}{\|x_k\|}, N(x_k)\right\rangle\right|.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{k}+a(n-r+1)\frac{\sigma_{r-1}(x_k)}{\|x_k\|} &> r[\sigma_r(x_k)]^2\\ &\quad-ar\left|\sigma_r(x_k)\left\langle\frac{x_k}{\|x_k\|}, N(x_k)\right\rangle\right|.\end{aligned}\tag{3.14}$$

Thus, using (3.13) and passing to the subsequence x_{k_l}, we obtain

$$\begin{aligned}
&\frac{1}{r}\lim_{l\to\infty}\left[\frac{1}{k_l}+a(n-r+1)\frac{\sigma_{r-1}(x_{k_l})}{\|x_{k_l}\|}\right]\\
&\geq \lim_{l\to\infty}[\sigma_r(x_{k_l})]^2-\sqrt{[\sigma_r(x_{k_l})]^4-(1-a^2)[\sigma_r(x_{k_l})]^2}\\
&=\lim_{l\to\infty}\alpha\left([\sigma_r(x_{k_l})]^2\right),
\end{aligned}\tag{3.15}$$

where $\alpha : [1-a^2, 1] \to \mathbb{R}$ is defined by

$$\alpha(t) = t - \sqrt{t^2 - (1-a^2)t}.$$

Since $\alpha'(t) < 0$, α is a decreasing function with minimum value at $t = 1$, $\alpha(1) = 1 - a$.

In order to conclude the proof, let us consider separately the cases of conditions (i) and (ii) of Theorem 1.1. First, let us assume condition (ii) of Theorem 1.1. This implies that σ_{r-1} is bounded and Σ^n is not necessarily proper.

Observe that the region $(\mathcal{C}_{V,a})^c$ is invariant by translations in the direction $-V$, i.e., if $\Sigma^n \subset (\mathcal{C}_{V,a})^c$, then $(\Sigma^n)_t := \Sigma^n - tV \subset (\mathcal{C}_{V,a})^c$ for any $t > 0$. Moreover, since Σ^n is a translating soliton with velocity vector V, each $(\Sigma^n)_t$ is also a translating soliton with the same velocity vector V. Then, considering Σ^n as one of the $(\Sigma^n)_t$, if necessary, we can assume that

$$\inf_{\Sigma^n}\|x\| \geq \frac{2a(n-r+1)\sup_{\Sigma^n}\sigma_{r-1}}{r(1-a)}.$$

This gives

$$\begin{aligned}
0 &\geq \lim_{l\to\infty}\alpha\left(\sigma_r(x_{k_l})^2\right)-\frac{1}{r}\lim_{l\to\infty}\left[\frac{1}{k_l}+a(n-r+1)\frac{\sigma_{r-1}(x_{k_l})}{\|x_{k_l}\|}\right]\\
&\geq 1-a-\frac{1}{r}\lim_{l\to\infty}\left[\frac{1}{k_l}+a(n-r+1)\frac{\sup_{\Sigma^n}\sigma_{r-1}}{\inf_{\Sigma^n}\|x\|}\right]\\
&\geq \frac{1-a}{2}\\
&> 0
\end{aligned}$$

which is an absurd.

Now, assume that condition (i) of Theorem 1.1 holds, i.e., Σ^n is properly immersed and

$$\limsup_{\delta_0(x)\to\infty}\frac{\sigma_{r-1}(x)}{\delta_0(x)} < \frac{r(1-a)}{a(n-r+1)},$$

(the difference between δ and δ_0 is a constant). This gives

$$\begin{aligned}0 &\geq \lim_{l\to\infty} \alpha\left(\sigma_r(x_{k_l})^2\right) - \frac{1}{r}\lim_{l\to\infty}\left[\frac{1}{k_l} + a(n-r+1)\frac{\sigma_{r-1}(x_{k_l})}{\|x_{k_l}\|}\right] \\ &\geq 1 - a - \frac{1}{r}\lim_{l\to\infty}\left[\frac{1}{k_l} + a(n-r+1)\frac{\sigma_{r-1}(x_{k_l})}{\|x_{k_l}\|}\right] \\ &> 0\end{aligned}$$

and we obtain an absurd again. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a properly immersed translating soliton of the r-mean curvature flow, with velocity V and operator P_{r-1} positive semidefinite satisfying (1.9). Suppose $\Sigma^n \subset \mathcal{H}_W$, where

$$\mathcal{H}_W = \{X \in \mathbb{R}^{n+1}; \langle X, W\rangle \leq 0,\ \langle V, W\rangle > 0,\ \|W\| = 1\}.$$

Let $c := \langle V, W\rangle > 0$ and define $\psi : \Sigma^n \to \mathbb{R}$ by $\psi(x) = \langle X(x), W\rangle$. By using (3.6) and (3.8) with W in the place of V, we have

$$\nabla\psi = W^\top$$

and

$$L_{r-1}\psi = r\sigma_r\langle N, W\rangle.$$

Since, by hypothesis, $\psi \leq 0$, by Theorem 2.1, p.11, there exists a sequence of points $\{x_k\}_k \subset \Sigma^n$ such that

$$\lim_{k\to\infty}\psi(x) = \sup_{\Sigma^n}\psi \leq 0, \tag{4.1}$$

$$\frac{1}{k} > \|\nabla\psi(x_k)\|^2 = \|W(x_k)^\top\|^2 = 1 - \langle W, N(x_k)\rangle^2 \geq 0, \tag{4.2}$$

and

$$\frac{1}{k} > L_{r-1}\psi(x_k) = r\sigma_r(x_k)\langle N(x_k), W\rangle. \tag{4.3}$$

Thus

$$\lim_{k\to\infty}\langle W, N(x_k)\rangle^2 = 1. \tag{4.4}$$

Since, for any orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n,

$$W = \sum_{i=1}^{n}\langle W, e_i\rangle e_i + \langle W, N\rangle N,$$

we have, by (4.4), that

$$\lim_{k\to\infty}\sum_{i=1}^{n}\langle W, e_i(x_k)\rangle^2 = 0.$$

Moreover,

$$\begin{aligned}\langle V, W\rangle &= \left\langle \sum_{i=1}^{n}\langle V, e_i\rangle e_i + \langle V, N\rangle N, \sum_{j=1}^{n}\langle W, e_j\rangle e_j + \langle W, N\rangle N\right\rangle\\ &= \sum_{i=1}^{n}\langle V, e_i\rangle\langle W, e_i\rangle + \langle V, N\rangle\langle W, N\rangle\end{aligned}$$

and the estimate $|\langle V, e_i(x_k)\rangle| \leq \|V\|\|e_i(x_k)\| \leq 1$, imply

$$\begin{aligned}0 &\geq \lim_{k\to\infty} L_{r-1}\psi(x_k)\\ &= r\lim_{k\to\infty}\langle N(x_k), V\rangle\langle N(x_k), W\rangle\\ &= r\left[\langle V, W\rangle - \lim_{k\to\infty}\sum_{i=1}^{n}\langle V, e_i(x_k)\rangle\langle e_i(x_k), W\rangle\right]\\ &= r\langle V, W\rangle = r\cdot c > 0,\end{aligned} \tag{4.5}$$

which is a contradiction. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3. This proof is inspired by Borbely's proof (see [14]) that complete minimal surfaces satisfying the Omori-Yau maximum principle cannot be in the intersection of two transversal vertical half-spaces.

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a translating soliton of the r-mean curvature flow with velocity V and $P_{r-1} \geq \varepsilon I$, $\varepsilon > 0$. Suppose, in addition, that σ_{r-1} satisfies (1.10). We are going to show that Σ^n cannot be contained in the intersection of two transversal vertical half-spaces.

Let

$$\mathcal{H}_i := \mathcal{H}_{(B_i, W_i)} := \{X \in \mathbb{R}^{n+1}; \langle X - B_i, W_i\rangle \geq 0\}, \quad i = 1, 2,$$

be two transversal vertical halfspaces. In order to simplify the proof, we can choose a system of coordinates of $\mathbb{R}^{n+1}$ such that

$$V = E_{n+1} = (0, \ldots, 0, 1),$$

$B_i = (0, \ldots, 0)$, $i = 1, 2$ and, by a rotation, we may consider

$$W_1 = (a, b, 0, \ldots, 0), \quad W_2 = (a, -b, 0, \ldots, 0),$$

where $a, b > 0$ and $a^2 + b^2 = 1$. In this system of coordinates, we have that

$$\mathcal{P}_i := \partial \mathcal{H}_i = \{X \in \mathbb{R}^{n+1}; \langle X, W_i \rangle = 0\}, \quad i = 1, 2.$$

Denote by $\mathcal{P}_i(R) := \mathcal{P}_i + RW_i = \{X \in \mathbb{R}^{n+1}; \langle X, W_i \rangle = R\}$ the hyperplanes parallel to $\mathcal{P}_i$ and by $\mathcal{L}(R)$ be the $(n-1)$-hyperplane $\mathcal{L}(R) = \mathcal{P}_1(R) \cap \mathcal{P}_2(R)$. Since the coordinates of each hyperplane $\mathcal{P}_i(R)$, $i = 1, 2$, satisfy the equation

$$ax_1 + (-1)^{i-1} bx_2 = R,$$

we have

$$\mathcal{L}(R) = \left\{ \left(\frac{R}{a}, 0, x_3, \ldots, x_{n+1} \right); (x_3, \ldots, x_{n+1}) \in \mathbb{R}^{n-1} \right\}.$$

Define

$$d_R(x) := \mathrm{dist}_{\mathbb{R}^{n+1}}(x, \mathcal{L}(R)) = \sqrt{\left(x_1 - \frac{R}{a} \right)^2 + x_2^2}, \tag{5.1}$$

for $x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, and consider the cylindrical region

$$\mathcal{D}(R) = \{x \in \mathbb{R}^{n+1}; d_R(x) \leq R\}.$$

The cylindrical region $\mathcal{D}(R)$ separates $(\mathcal{H}_1 \cap \mathcal{H}_2) \setminus \mathcal{D}(R)$ into two regions, one with $d_R(x)$ bounded (which we denote by $\mathcal{V}_R$) and another where $d_R(x)$ is unbounded (see Figure 2).

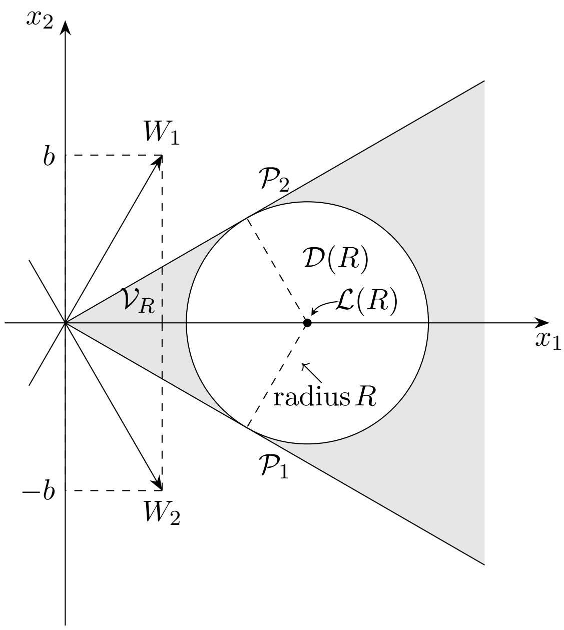

FIGURE 2. Regions in the intersection of vertical halfspaces

We assume, by contradiction, that there exists a complete, properly immersed, translating soliton of the r-mean curvature flow, with $P_{r-1} \geq \varepsilon I$, for $\varepsilon > 0$, such that σ_{r-1} satisfies (1.8) and $\Sigma^n \subset \mathcal{H}_1 \cap \mathcal{H}_2$.

Since $\mathcal{V}_R \to \mathcal{H}_1 \cap \mathcal{H}_2$ when $R \to \infty$, we can assume $R > 0$ large enough such that $\Sigma^n \cap \mathcal{V}_R \neq \emptyset$. Let us denote by $\overline{\nabla}$ the gradient and connection of $\mathbb{R}^{n+1}$ and by $\overline{\text{Hess}}$ the hessian quadratic form of $\mathbb{R}^{n+1}$. The following facts are straightforward calculations about the function $d(x) := d_R(x)$, for $x \in (\mathcal{H}_1 \cap \mathcal{H}_2) \backslash \mathcal{L}(R)$:

(i) $\overline{\text{Hess}}\ d(x)(\overline{\nabla} d, \overline{\nabla} d) = 0$ which implies that $\overline{\nabla} d$ is an eigenvector of $\overline{\text{Hess}}\ d$ with eigenvalue zero;

(ii) Since $d(x)$ does not depend on $x_3, \ldots, x_{n+1}$, the vectors $E_3, \ldots, E_{n+1}$ of the canonical basis of $\mathbb{R}^{n+1}$ are also eigenvectors of $\overline{\text{Hess}}\ d$ with eigenvalue zero;

(iii) The last eigenvector of $\overline{\text{Hess}}\ d$ is given by

$$\chi = \left(-\frac{\partial d}{\partial x_2}, \frac{\partial d}{\partial x_1}, \vec{\mathbf{0}} \right) = \left(-\frac{x_2}{d}, \frac{x_1 - R/a}{d}, \vec{\mathbf{0}} \right), \quad \vec{\mathbf{0}} \in \mathbb{R}^{n-1}.$$

For this eigenvector, the eigenvalue is $1/d$.

These facts imply that the set $\{\overline{\nabla} d, \chi, E_3, \ldots, E_{n+1}\}$ is an orthonormal frame of $\mathbb{R}^{n+1}$ given by eigenvectors of $\overline{\text{Hess}}\ d$. This implies that any vector field $Y \in \mathbb{R}^{n+1}$ can be written as

$$Y = \langle Y, \overline{\nabla} d \rangle \overline{\nabla} d + \langle Y, \chi \rangle \chi + \sum_{j=3}^{n+1} \langle Y, E_j \rangle E_j. \tag{5.2}$$

Moreover, for any $Y, Z \in \mathbb{R}^{n+1}$,

$$\overline{\text{Hess}}\ d(Y, Z) = \frac{1}{d} \langle Y, \chi \rangle \langle Z, \chi \rangle. \tag{5.3}$$

Now, let $f : \Sigma^n \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} d_R(x), & \text{if} \quad x \in \Sigma^n \cap \mathcal{V}_R, \\ R, & \text{if} \quad x \in \Sigma^n \backslash \mathcal{V}_R. \end{cases}$$

Notice that f is continuous and differentiable everywhere, except for points of $\Sigma^n \cap (\partial \mathcal{V}_R \cap \partial \mathcal{D}(R))$. Moreover $R \leq f(x) < R/a$ (since $0 < a < 1$) and

$$R < \sup_{\Sigma^n} f \leq R/a < \infty,$$

since $\Sigma^n \cap \mathcal{V}_R \neq \emptyset$ and $f(x) > R$ for $x \in \Sigma^n \cap \mathcal{V}_R$ (see Figure 2). The following calculations will be carried out at points $x \in \Sigma^n \cap \mathcal{V}_R$ where the function d_R is differentiable. Since

$$\nabla f = (\overline{\nabla} d)^{\top} = \overline{\nabla} d - (\overline{\nabla} d)^{\perp} = \overline{\nabla} d - \langle \overline{\nabla} d, N\rangle N \tag{5.4}$$

and, by replacing N in the place of Y in (5.2),

$$\begin{aligned} 1 &= \langle \chi, N\rangle^2 + \langle \overline{\nabla} d, N\rangle^2 + \sum_{j=3}^{n+1} \langle E_j, N\rangle^2 \\ &\geq \langle \chi, N\rangle^2 + \langle \overline{\nabla} d, N\rangle^2, \end{aligned}$$

we have

$$\|\nabla f\| = \sqrt{\|\overline{\nabla} d\|^2 - \langle \overline{\nabla} d, N\rangle^2} = \sqrt{1 - \langle \overline{\nabla} d, N\rangle^2} \geq |\langle \chi, N\rangle|, \tag{5.5}$$

provided $\|\overline{\nabla} d\| = 1$.

If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of Σ^n, given by eigenvectors of the second fundamental form, then

$$\begin{aligned} \sum_{i=1}^{n} \overline{\text{Hess}}\ d(e_i, P_{r-1}(e_i)) &= \frac{1}{d} \sum_{i=1}^{n} \langle e_i, \chi\rangle \langle P_{r-1}(e_i), \chi\rangle \\ &= \frac{1}{d} \sum_{i=1}^{n} \lambda_i \langle e_i, \chi\rangle^2, \end{aligned} \tag{5.6}$$

where λ_i is the eigenvalue of P_{r-1} relative to the eigenvector e_i, $i = 1, \ldots, n$. Since, by hypothesis, $\lambda_i \geq \varepsilon > 0$ and

$$\sum_{i=1}^{n} \langle e_i, \chi\rangle^2 = 1 - \langle N, \chi\rangle^2,$$

we have

$$\sum_{i=1}^{n} \overline{\text{Hess}}\ d(e_i, P_{r-1}(e_i)) \geq \varepsilon \left(\frac{1 - \langle N, \chi\rangle^2}{d} \right). \tag{5.7}$$

On the other hand, for any $Y, Z \in T\Sigma^n$, and for points $x \in \Sigma^n \cap \mathcal{V}_R$,

$$\text{Hess}\ f(Y, Z) = \overline{\text{Hess}}\ d(Y, Z) + \langle \overline{\nabla} d, N\rangle \langle A(X), Y\rangle,$$

where A is the shape operator of Σ^n. This gives, for any orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ of $T\Sigma^n$,

$$
\begin{aligned}
L_{r-1}f &= \operatorname{trace}(\operatorname{hess} f \circ P_{r-1}) = \sum_{i=1}^{n} \operatorname{Hess} f(e_i, P_{r-1}(e_i)) \\
&= \sum_{i=1}^{n} \overline{\operatorname{Hess}}\, d(e_i, P_{r-1}(e_i)) + \langle \overline{\nabla} d, N\rangle \sum_{i=1}^{n} \langle A(e_i), P_{r-1}(e_i)\rangle \\
&= \sum_{i=1}^{n} \overline{\operatorname{Hess}}\, d(e_i, P_{r-1}(e_i)) + r\sigma_r \langle \overline{\nabla} d, N\rangle \\
&= \sum_{i=1}^{n} \overline{\operatorname{Hess}}\, d(e_i, P_{r-1}(e_i)) + r\langle E_{n+1}, N\rangle \langle \overline{\nabla} d, N\rangle,
\end{aligned} \tag{5.8}
$$

where we used that

$$\sum_{i=1}^{n} \langle A(e_i), P_{r-1}(e_i)\rangle = \operatorname{trace}(AP_{r-1}) = r\sigma_r$$

and that $\sigma_r = \langle E_{n+1}, N\rangle$. Combining (5.8) with (5.7), we obtain

$$L_{r-1}f \geq \varepsilon \left(\frac{1 - \langle \chi, N\rangle^2}{d} \right) + r\langle E_{n+1}, N\rangle \langle \overline{\nabla} d, N\rangle. \tag{5.9}$$

Since f is bounded, $f(x) = R$ for $x \in \Sigma^n \backslash \mathcal{V}_R$ and $\sup_{\Sigma^n} f > R$, by Theorem 2.1, p.11, there exists a sequence $\{x_k\}_k$ of points in $\Sigma^n \cap \mathcal{V}_R$ such that
(5.10)

$$\lim_{k\to\infty} f(x_k) = \sup_{\Sigma^n} f,\ \lim_{k\to\infty} \|\nabla f(x_k)\| = 0 \quad \text{and} \quad \limsup_{k\to\infty} L_{r-1}f(x_k) \leq 0.$$

Now, let us analyze the last term of (5.9). First notice that, since

$$\langle E_{n+1}, \overline{\nabla} d\rangle = 0,$$

we have

$$
\begin{aligned}
\langle E_{n+1}, N\rangle^2 &= \langle E_{n+1}, N \pm \overline{\nabla} d\rangle^2 \\
&\leq \|N \pm \overline{\nabla} d\|^2 \\
&= 2(1 \pm \langle N, \overline{\nabla} d\rangle).
\end{aligned} \tag{5.11}
$$

Using (5.5) and the second limit in (5.10), we obtain that

$$\lim_{k\to\infty} \langle N(x_k), \overline{\nabla} d(x_k)\rangle^2 = 1. \tag{5.12}$$

It means that there exists at least one subsequence (which we also denote by $\{x_k\}_k$ to simplify the notation) such that

$$\lim_{k\to\infty} \langle N(x_k), \overline{\nabla} d(x_k)\rangle = 1 \quad \text{or} \quad \lim_{k\to\infty} \langle N(x_k), \overline{\nabla} d(x_k)\rangle = -1.$$

In both cases, we can use (5.11) to obtain

$$\lim_{k\to\infty}\langle E_{n+1}, N(x_k)\rangle = 0. \tag{5.13}$$

This implies

$$\lim_{k\to\infty}\langle E_{n+1}, N(x_k)\rangle\langle N(x_k), \overline{\nabla} d(x_k)\rangle = 0. \tag{5.14}$$

On the other hand, by (5.5) and (5.10), we have

$$\lim_{k\to\infty}\langle N(x_k), \chi(x_k)\rangle = 0. \tag{5.15}$$

Thus, by using (5.9), (5.10), and (5.15), we have

$$\begin{aligned}
0 &\geq \limsup_{k\to\infty} L_{r-1} f(x_k) \\
&\geq \varepsilon \limsup_{k\to\infty}\left(\frac{1-\langle N(x_k), \chi(x_k)\rangle^2}{d(x_k)}\right) \\
&\quad + r\limsup_{k\to\infty}\langle E_{n+1}, N(x_k)\rangle\langle N(x_k), \overline{\nabla} d(x_k)\rangle \\
&\geq \frac{\varepsilon a}{R} > 0,
\end{aligned}$$

which is a contradiction. Here we used that $d(x) \leq R/a$. Thus, there is no such translating soliton. □

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