RIGIDITY OF COMPLETE SELF-SHRINKERS WHOSE TANGENT PLANES OMIT A NONEMPTY SET

HILÁRIO ALENCAR, MANUEL CRUZ & GREGÓRIO SILVA NETO

Abstract. In this paper we prove rigidity results for the sphere, the plane and the right circular cylinder as the only self-shrinkers satisfying a new geometric assumption, namely the union of all tangent affine submanifolds of a complete self-shrinker omits a non-empty set of the Euclidean space. This assumption lead us to a new class of submanifolds, different from those with polynomial volume growth or the proper ones (which was proved to be equivalent by Cheng and Zhou, see [6]). In fact, in the last section, we present an example of a non proper surface whose tangent planes omit the interior of a right circular cylinder, which proves that these classes are distinct from each other.

1. Introduction

A n-dimensional submanifold $X : \Sigma^n \to \mathbb{R}^{n+k}, n \geq 2,\ k \geq 1$, is called a self-shrinker if it satisfies

$$\mathbf{H} = -\frac{1}{2}X^{\perp},$$

where $\mathbf{H} = \sum_{i=1}^{n} \alpha(e_i, e_i)$ is the mean curvature vector field of Σ^n and $X^{\perp}$ is the part of X normal to Σ^n.

Self-shrinkers are self-similar solutions of the mean curvature flow and plays an important role in the study of this flow since they are type I singularities of the flow, see [7]. The simplest examples of self-shrinkers are the round spheres, planes and cylinders. Moreover, there are many results which present these examples as the only self-shrinkers satisfying some geometric restrictions, see [7], [2], [1], [9], and [10]. In commom, all these results

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have the assumption that, when the self-shrinker is not compact, it must have polynomial volume growth or it must be proper. In [6], Cheng and Zhou proved that a self-shrinker has polynomial volume growth if and only if it is proper. Recently, joinly with Vieira, see [4], they generalized this result for submanifolds with bounded weighted mean curvature in a wide class of shrinking gradient Ricci solitons, which includes the Gaussian soliton. In particular, Cheng-Vieira-Zhou result gives that, for a surface with bounded $\mathbf{H}+\frac{1}{2}X^{\perp}$, polynomial volume growth is equivalent to the properness of the submanifold (see Theorems 1.3 and 1.4 of [4]).

In this paper, we prove the rigidity of the sphere, the cylinders and the affine subspaces passing through the origin as the only self-shrinkers under a new geometric assumption we describe below. Here and elsewhere, we identify the tangent spaces $T_p\Sigma^n$ with the affine subspace $X(p)+dX_p(T_p\Sigma^n)$, tangent to $X(\Sigma)$ at $X(p)$.

Let us denote by

$$W=\mathbb{R}^{n+k}\backslash\bigcup_{p\in\Sigma^n}T_p\Sigma^n$$

the set omitted by the union of the affine subspaces tangent to $X(\Sigma^n)\subset\mathbb{R}^{n+k}$. Here, we purpose to classify the self-shrinkers with nonempty W. It is important to remark that submanifolds with $W\neq\emptyset$ is a class of submanifolds distinct from those with polynomial of volume growth or those which are proper. In fact, in section 4 we give an example of a non proper surface (and, thus, with volume growth bigger than polynomial, by Cheng-Vieira-Zhou [4] results) such that $W=\mathbb{D}^2\times\mathbb{R}$, and thus, nonempty. Here $\mathbb{D}^2=\{(x,y)\in\mathbb{R}^2; x^2+y^2\leq 1\}$ is the closed unit disk.

First, we present two rigidity results for complete n-dimensional self-shrinkers in $\mathbb{R}^{n+1}$ with open and nonempty W.

Theorem 1.1. *Let Σ^n be a complete, n-dimensional, self-shrinker of $\mathbb{R}^{n+1}$. If the set W is open and nonempty, and the squared matrix norm $|A|^2$ of the second fundamental form A of Σ^n satisfies*

$$|A|^2\leq\frac{1}{2},$$

then $\Sigma^n=\mathbb{S}^p(\sqrt{2p})\times\mathbb{R}^{n-p}$, $0\leq p\leq n$.

If $|A|^2\geq 1/2$ and, additionally, we assume the mean curvature $H\geq 0$, we have

Theorem 1.2. *Let Σ^n be a complete, n-dimensional, self-shrinker of $\mathbb{R}^{n+1}$ with mean curvature $H \geq 0$. If the set W is open, nonempty, and the squared matrix norm $|A|^2$ of the second fundamental form A of Σ^n satisfies*

$$|A|^2 \geq \frac{1}{2},$$

then $\Sigma^n = \mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$.

Remark 1.1. The bound $1/2$ seens natural for self-shrinkers. In [2], Cao and Li proved that the only complete n-dimensional self-shrinkers of $\mathbb{R}^{n+k}$ with polynomial volume growth and such that $|A|^2 \leq 1/2$ are $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$. Cheng and Peng, see [3], and Rimoldi, see [17], with the aim to remove the hypothesis of polyminal volume growth, proved that the only self-shrinker of $\mathbb{R}^{n+1}$ with $\sup_{\Sigma^n} |A|^2 = 1/2$ (but with $|A|^2 < 1/2$) is a hyperplane. On the other hand, there are other rigidity results where the bound $|A|^2 \geq 1/2$ appears see, for example, [15], [9], [5], and [19]. Again, in common, all the last four references assumes polynomial volume growth or the immersion is proper.

If Σ^2 has dimension two, then we can consider arbitrary codimension. We will assume further that $\mathbf{H} \neq 0$ and that $\mathbf{H}/|\mathbf{H}|$ is parallel at the normal bundle.

Theorem 1.3. *Let Σ^2 be a complete, two-dimensional, self-shrinker of $\mathbb{R}^{2+k}$, $k \geq 2$, with mean curvature vector $\mathbf{H} \neq 0$ and such that $\mathbf{H}/|\mathbf{H}|$ is parallel at the normal bundle. If the set W is open and nonempty and the squared matrix norm $|A|^2$ of the second fundamental form A of Σ^n, relative to $\mathbf{H}/|\mathbf{H}|$, satisfies one of the following conditions:*

i) $|A|^2 \leq 1/2$;
ii) $|A|^2 \geq 1/2$;

then $\Sigma^2 = \mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{2-p}$, $1 \leq p \leq 2$.

Remark 1.2. If we assume that Σ^2 is compact without boundary in Theorem 1.3, then the hypothesis that W is open can be removed. We point out that Smoczyk, see [18], proved that the only compact self-shrinkers, without boundary, of $\mathbb{R}^{n+p}$ with $\mathbf{H} \neq 0$ and $\mathbf{H}/|\mathbf{H}|$ parallel in the normal bundle are minimal surfaces of the sphere $\mathbb{S}^{n+p-1}(\sqrt{2n})$.

Remark 1.3. The study of submanifolds of the Euclidean space with non-empty W started with Halpern, see [12], who proved that compact and oriented hypersurfaces of the Euclidean space have nonempty W if and only if it is embedded, diffeomorphic to the sphere and it is the boundary of a star-shaped domain of $\mathbb{R}^{n+1}$. Therefore, since the only self-shrinker with these characteristics is the round sphere of radius $\sqrt{2n}$ (see [13]), the case of compact self-shrinkers of codimension one with nonempty W is completely solved.

Remark 1.4. Drugan and Kleene in [11] proved the existence of infinitely many rotational self-shrinkers of each topological type of $\mathbb{S}^n$, $\mathbb{S}^{n-1}\times\mathbb{S}^1$, $\mathbb{R}^n$ and $\mathbb{S}^{n-1}\times\mathbb{R}$. Analyzing geometrically the picture of the profile curves presented there, we can see that the rotational self-shrinkers obtained by the rotation of those profiles curves have empty W. We also remark that all these examples are not embedded since Kleene and Møller, see [14], proved that the sphere of radius $\sqrt{2n}$, the plane, and the right cylinder of radius $\sqrt{2(n-1)}$ are the only embedded rotational self-shrinkers of their respective topological type.

We conclude this paper with a non existence result for self-expanders. Recall that a n-dimensional submanifold $X:\Sigma^n\to\mathbb{R}^{n+k}$ is called a self-expander if it satisfies

$$\mathbf{H}=\frac{1}{2}X^{\perp}.$$

Theorem 1.4. *There is no complete, non compact, n-dimensional self-expanders of $\mathbb{R}^{n+k}$, $k\geq 1$, with mean curvature vector $\mathbf{H}\neq 0$, $\mathbf{H}/|\mathbf{H}|$ parallel in the normal bundle, and such that the set W is open and nonempty.*

Remark 1.5. It is well known, see [2], that there is no compact self-expanders in $\mathbb{R}^{n+k}$. Thus, Theorem 1.4 does not make sense for Σ^n compact.

2. Preliminaries

Let $i:\Sigma^n\to M^{n+k},\ n\geq 2,\ k\geq 1$, be an isometric immersion, where Σ^n and M^{n+k} are Riemannian manifolds and the superscripts denote the dimension. Denote by ∇ and $\overline{\nabla}$ be the connections of Σ^n and M^{n+k}, respectively. We assume here that the immersion admits a conformal vector field, i.e., a vector field $X\in TM$ such that

$$\overline{\nabla}_Y X=\varphi Y, \tag{2.1}$$

for some smooth function $\varphi: M^{n+k}\to\mathbb{R}$, called conformal factor of X, and for every $Y\in T\Sigma^n$. Decompose X as

$$X=X^{\top}+X^{\perp},$$

where $X^\top \in T\Sigma^n$ and $X^\perp \in (T\Sigma^n)^\perp$. Here $(T\Sigma^n)^\perp$ is the normal bundle of the immersion such that $T\Sigma^n \oplus (T\Sigma^n)^\perp = TM^{n+k}$.

If the codimension is one, then we have $X^\perp = \langle X, N\rangle N$, where N is the globally defined unitary normal vector field. If the codimension is at least two, suppose further that $X^\perp \neq 0$. In both cases we can write

$$X = X^\top + f\eta, \tag{2.2}$$

for $f = \langle X, \eta\rangle$, where $\eta = N$ if the codimension is one and $\eta = X^\perp/|X^\perp|$ if the codimension is at least two.

The immersion satisfies

$$\overline{\nabla}_U V = \nabla_U V + \alpha(U, V) \text{ and } \overline{\nabla}_U \eta = -AU + \nabla^\perp_U \eta, \tag{2.3}$$

where $\langle AU, V\rangle = \langle \alpha(U, V), \eta\rangle$, α is the second fundamental form of the immersion, and $\nabla^\perp$ denotes the normal connection at the normal bundle $(T\Sigma^n)^\perp$.

The next proposition contains the basic calculations needed to prove the main theorems of this paper.

Proposition 2.1. *Let M^{n+k} be a $(n+k)$-dimensional Riemannian manifold which admits a conformal vector field X with conformal factor φ. Let Σ^n be a submanifold of M^{n+k} and $\{\eta, \eta_2, \ldots, \eta_k\}$ be an orthonormal frame of the normal bundle $(T\Sigma^n)^\perp \subset TM^{n+k}$, where, for $k = 1$, $\eta = N$, the globally defined unitary normal vector field, and for $k \geq 2$, we assume that $X^\perp \neq 0$ and take $\eta = X^\perp/|X^\perp|$. If $f = \langle X, \eta\rangle$, then*

$$\begin{aligned} &\Delta f + \varphi(\operatorname{trace} A) + f|A|^2 + \langle X^\top, \operatorname{grad}(\operatorname{trace} A)\rangle \\ &= -\sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^\top) + \sum_{\beta=2}^{k} s_{1\beta}(X^\top)(\operatorname{trace} A_\beta) \\ &\quad + \sum_{i=1}^{n} \langle \overline{R}(e_i, X^\top)e_i, \eta\rangle. \end{aligned} \tag{2.4}$$

Here, A and A_β are the shape operators relative to the normals η and η_β, $\beta \in \{2, \ldots, k\}$, respectively, $|A|^2 = \operatorname{trace}(A^2)$ is the matrix norm of A, $s_{1\beta}(X) = \langle \nabla^\perp_X \eta, \eta_\beta\rangle$, $\overline{R}$ is the curvature tensor of M^{n+k}, and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of Σ^n.

If the immersion has codimension one (i.e., $k = 1$), then

$$\Delta f + \varphi H + f|A|^2 + \langle X^\top, \operatorname{grad} H\rangle = \sum_{i=1}^{n} \langle \overline{R}(e_i, X^\top)e_i, \eta\rangle, \tag{2.5}$$

where $H = \operatorname{trace} A$ is the mean curvature of Σ^n. In particular, if the Ricci curvature of M^{n+1} is constant (i.e., M^{n+1} is an Einstein space), then

$$\Delta f + \varphi H + f|A|^2 + \langle X^\top, \operatorname{grad} H\rangle = 0. \tag{2.6}$$

Proof. Let $U \in T\Sigma^n$. Since, using (2.3),

$$\begin{aligned} \varphi U = \overline{\nabla}_U X &= \overline{\nabla}_U X^\top + (Uf)\eta + f\overline{\nabla}_U \eta \\ &= \nabla_U X^\top + \alpha(X^\top, U) + (Uf)\eta - fAU + f\nabla^\perp_U \eta, \end{aligned}$$

we have, taking the tangent and the normal parts,

$$\varphi U = \nabla_U X^\top - fAU \tag{2.7}$$

and

$$\alpha(X^\top, U) + (Uf)\eta + f\nabla^\perp_U \eta = 0. \tag{2.8}$$

From (2.7) we have

$$\nabla_U X^\top = (\varphi I + fA)U, \tag{2.9}$$

which implies

$$\operatorname{div} X^\top = n\varphi + f(\operatorname{trace} A), \tag{2.10}$$

where $\operatorname{div} X^\top$ is the divergence of $X^\top$ in Σ^n. From (2.8) we obtain

$$Uf = -\langle \alpha(X^\top, U), \eta\rangle, \tag{2.11}$$

since $\langle \nabla^\perp_U \eta, \eta\rangle = 0$. Therefore,

$$\operatorname{grad} f = -AX^\top. \tag{2.12}$$

Let $\{\eta_1 = \eta, \eta_2, \dots, \eta_k\}$ be an orthonormal frame of $(T\Sigma^n)^\perp$ and write

$$\nabla^\perp_U \eta = \sum_{\beta=2}^{k} s_{1\beta}(X)\eta_\beta, \text{ where } s_{1\beta}(X) = \langle \nabla^\perp_X \eta, \eta_\beta\rangle.$$

Taking the inner product of (2.8) with η_β, we have

$$\langle \alpha(X^\top, U), \eta_\beta\rangle + f s_{1\beta}(U) = 0$$

i.e.,

$$f s_{1\beta}(U) = -\langle A_\beta X^\top, U\rangle, \tag{2.13}$$

where $\langle A_\beta U, V\rangle = \langle \alpha(U,V), \eta_\beta\rangle$.

Let us calculate the Laplacian of f. Since, by (2.11), $Uf = -\langle AX^\top, U\rangle$, and using (2.9), we obtain

$$\begin{aligned} U(Uf) &= -U\langle AX^\top, U\rangle = -U\langle X^\top, AU\rangle \\ &= -\langle \nabla_U X^\top, AU\rangle - \langle X^\top, \nabla_U(AU)\rangle \\ &= -\varphi\langle U, AU\rangle - f\langle AU, AU\rangle - \langle X^\top, \nabla_U(AU)\rangle \end{aligned}$$

and $(\nabla_U U)f = -\langle AX^\top, \nabla_U U\rangle = -\langle X^\top, A(\nabla_U U)\rangle$. This implies

$$\operatorname{Hess} f(U,U) = -\varphi\langle U, AU\rangle - f\langle AU, AU\rangle - \langle X^\top, (\nabla_U A)(U)\rangle,$$

where $(\nabla_U A)(V) = \nabla_U AV - A(\nabla_U V)$. Taking the trace, we have

$$\Delta f = -\varphi(\operatorname{trace} A) - f(\operatorname{trace}(A^2)) - \sum_{i=1}^n \langle X^\top (\nabla_{e_i} A)(e_i)\rangle,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of $T\Sigma^n$. On the other hand, the Codazzi equation

$$\begin{aligned} \langle \overline{R}(U,V)W, \eta\rangle &= \langle (\nabla_V A)(U) - (\nabla_U A)(V), W\rangle \\ &\quad + \langle \alpha(V,W), \nabla^\perp_U \eta\rangle - \langle \alpha(U,W), \nabla^\perp_V \eta\rangle \end{aligned} \tag{2.14}$$

and

$$\langle \alpha(V,W), \nabla^\perp_U \eta\rangle = \sum_{\beta=2}^k s_{1\beta}(U)\langle \alpha(V,W), \eta_\beta\rangle = \sum_{\beta=2}^k s_{1\beta}(U)\langle A_\beta V, W\rangle$$

give

$$\begin{aligned} (\nabla_U A)(V) &= (\nabla_V A)(U) + \sum_{\beta=2}^k [s_{1\beta}(U)A_\beta V - s_{1\beta}(V)A_\beta U] \\ &\quad + \sum_{i=1}^n \langle \overline{R}(U,V)\eta, e_k\rangle e_k. \end{aligned} \tag{2.15}$$

Since A is symmetric, $\nabla_U A$ is symmetric also, and moreover

$$\operatorname{trace}(\nabla_U A) = U(\operatorname{trace} A).$$

These equations give

$$
\begin{aligned}
\sum_{i=1}^{n}\langle X^{\top}, (\nabla_{e_i}A)(e_i)\rangle = \sum_{i=1}^{n}\langle(\nabla_{e_i}A)(X^{\top}), e_i\rangle &= \sum_{i=1}^{n}\langle(\nabla_{X^{\top}}A)(e_i), e_i\rangle\\
&\quad + \sum_{\beta=2}^{k}\sum_{i=1}^{n}[s_{1\beta}(e_i)\langle A_\beta X^{\top}, e_i\rangle - s_{1\beta}(X^{\top})\langle A_\beta e_i, e_i\rangle]\\
&\quad + \sum_{i=1}^{n}\langle\overline{R}(e_i, X^{\top})\eta, e_i\rangle\\
&= \operatorname{trace}(\nabla_{X^{\top}}A) + \sum_{\beta=2}^{k}\sum_{i=1}^{n} s_{1\beta}(e_i)\langle A_\beta X^{\top}, e_i\rangle\\
&\quad - \sum_{\beta=2}^{k} s_{1\beta}(X^{\top})(\operatorname{trace} A_\beta) + \sum_{i=1}^{n}\langle\overline{R}(e_i, X^{\top})\eta, e_i\rangle\\
&= \langle X^{\top}, \operatorname{grad}(\operatorname{trace} A)\rangle + \sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^{\top})\\
&\quad - \sum_{\beta=2}^{k} s_{1\beta}(X^{\top})(\operatorname{trace} A_\beta) + \sum_{i=1}^{n}\langle\overline{R}(e_i, X^{\top})\eta, e_i\rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\Delta f &= -\varphi(\operatorname{trace} A) - f|A|^2 - \langle X^{\top}, \operatorname{grad}(\operatorname{trace} A)\rangle\\
&\quad - \sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^{\top}) + \sum_{\beta=2}^{k} s_{1\beta}(X^{\top})(\operatorname{trace} A_\beta) + \sum_{i=1}^{n}\langle\overline{R}(e_i, X^{\top})e_i, \eta\rangle,
\end{aligned}
\tag{2.16}
$$

where $|A|^2 = \operatorname{trace}(A^2)$ is the matrix norm of A. □

In the next consequence of Proposition 2.1, let us assume that there exists $\varepsilon \in \mathbb{R}$ such that, restricted to Σ^n,

$$
\mathbf{H} = \varepsilon X^{\perp}, \tag{2.17}
$$

where $\mathbf{H} = \sum_{i=1}^{n}\alpha(e_i, e_i)$ is the mean curvature vector field of Σ^n in M^{n+k}. If $M^{n+k} = \mathbb{R}^{n+k}$, then Σ^n is a mean curvature flow soliton, which is called a self-shrinker, if $\varepsilon < 0$, and a self-expander, if $\varepsilon > 0$. Here, we will adopt one of the canonical normalizations, considering $\varepsilon = -\frac{1}{2}$ for self-shrinkers and $\varepsilon = \frac{1}{2}$ for self-expanders.

If Σ^n is submanifold of M^{n+k} satisfying (2.17), then

$$
\operatorname{trace} A = \varepsilon f \ \text{ and } \ \operatorname{trace} A_\beta = 0. \tag{2.18}
$$

Let us define the elliptic operator $\mathcal{L}f$ by

$$\mathcal{L}f = \Delta f + \varepsilon\langle X, \operatorname{grad} f\rangle. \tag{2.19}$$

The next result is a direct consequence of Proposition 2.1, and gives us the main equations to prove our results.

Corollary 2.1. *Let M^{n+k} be a $(n+k)$-dimensional Riemannian manifold which admits a conformal vector field X with conformal factor φ. Let Σ^n be a submanifold of M^{n+k} such that the mean curvature vector $\mathbf{H}$ of Σ^n satisfies $\mathbf{H} = \varepsilon X^\perp$ for some $\varepsilon \in \mathbb{R}$, and $\{\eta, \eta_2, \dots, \eta_k\}$ be an orthonormal frame of the normal bundle $(T\Sigma^n)^\perp \subset TM^{n+k}$, where, for $k = 1$, $\eta = N$, the globally defined unitary normal vector field, and for $k \geq 2$, we assume that $X^\perp \neq 0$ and take $\eta = X^\perp/|X^\perp|$. If $f = \langle X, \eta\rangle$, then*

$$\mathcal{L}f + (|A|^2 + \varepsilon\varphi)f = -\sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^\top) + \sum_{i=1}^{n}\langle \overline{R}(e_i, X^\top)e_i, \eta\rangle. \tag{2.20}$$

Here, A and A_β are the shape operators relative to the normals η and η_β, $\beta \in \{2, \dots, k\}$, respectively, $s_{1\beta}(U) = \langle \nabla^\perp_U \eta, \eta_\beta\rangle$, $\overline{R}$ is the curvature tensor of M^{n+k}, and $\{e_1, \dots, e_n\}$ is an orthonormal frame of $T\Sigma^n$. Moreover, if $f \neq 0$, then

$$\mathcal{L}f + (|A|^2 + \varepsilon\varphi)f = \frac{1}{f}\sum_{\beta=2}^{k}|A_\beta X^\top|^2 + \sum_{i=1}^{n}\langle \overline{R}(e_i, X^\top)e_i, \eta\rangle. \tag{2.21}$$

In particular, if M^{n+k} has constant sectional curvature and $\nabla^\perp\eta = 0$, or the immersion has codimension one and M^{n+1} is Einstein, then

$$\mathcal{L}f + (|A|^2 + \varepsilon\varphi)f = 0. \tag{2.22}$$

Proof. By using (2.18) and (2.19) in (2.4), p.5, we obtain (2.20). Equation (2.21) comes from replacing (2.13) in the first term of the right hand side of (2.20). To prove (2.22), notice that, if M^{n+k} has constant sectional curvature κ_0, then

$$\langle \overline{R}(e_i, X^\top)e_i, \eta\rangle = \kappa_0(\langle e_i, e_i\rangle\langle X^\top, \eta\rangle - \langle X^\top, e_i\rangle\langle \eta, e_i\rangle) = 0,$$

since $\langle X^\top, \eta\rangle = 0 = \langle \eta, e_i\rangle$. Moreover, if $\nabla^\perp\eta = 0$, then $s_{1\beta} \equiv 0$ for every $\beta \in \{2, \dots, k\}$, i.e., $A_\beta X^\top = 0$. On the other hand, if $\mathrm{Ric}_M = \lambda\langle\cdot,\cdot\rangle$, $\lambda \in \mathbb{R}$, then

$$\begin{aligned}\sum_{i=1}^{n}\langle \overline{R}(e_i, X^\top)e_i, \eta\rangle &= \mathrm{Ric}_M(X^\top, \eta) - \langle \overline{R}(\eta, X^\top)\eta, \eta\rangle \\ &= \mathrm{Ric}_M(X^\top, \eta) = \lambda\langle X^\top, \eta\rangle = 0.\end{aligned}$$

□

In order to prove our results, we will also need the classical Hopf maximum principle for elliptic operators:

Lemma 2.1 (Hopf's maximum principle, see [16]). *Let*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

be a strictly elliptic differential operator defined in a open set $\Omega \subset \mathbb{R}^n$.

(i) *If* $c = 0$, $Lu \geq 0$ *(resp.* $Lu \leq 0$*) and there exists* $\max_\Omega u$ *(resp.* $\min_\Omega u$*), then* u *is constant.*

(ii) *If* $c \leq 0$, $Lu \geq 0$ *(resp.* $Lu \leq 0$*) and there exists* $\max_\Omega u \geq 0$ *(resp.* $\min_\Omega u \leq 0$*), then* u *is constant.*

(iii) *Independently of the signal of* c, *if* $Lu \geq 0$ *(resp.* $Lu \leq 0$*) and* $\max_\Omega u = 0$ *(resp.* $\min_\Omega u = 0$*), then* u *is constant.*

3. Proof of the main theorems

Now we are ready to proof our main theorems.

Proof of Theorem 1.1. In $\mathbb{R}^{n+1}$, the position vector is a conformal vector field with conformal factor $\varphi = 1$. Since Σ^n is a self-shrinker, we have

$$H = -\frac{1}{2}\langle X, N\rangle = -\frac{1}{2}f, \tag{3.1}$$

where N is a unitary normal vector field. Since the codimension is one and Σ^n is a self-shrinker, using Equation (2.22) of Proposition 2.1 for $\varepsilon = -1/2$, we obtain

$$\mathcal{L}f + \left(|A|^2 - \frac{1}{2}\right) f = 0. \tag{3.2}$$

Since $W \neq \emptyset$, there exists $p_0 \in W$, i.e., $p_0 \notin \bigcup_{p\in\Sigma^n} T_p\Sigma^n$, which implies that $p - p_0 \notin T_p\Sigma^n$ for every $p \in \Sigma^n$. This implies that the angle between $p - p_0$ and $N(p)$ is never $\pi/2$ (see Figure 1). In fact, otherwise, $p - p_0 \perp N(p)$ which would imply $p - p_0 \in T_p\Sigma$, a contradiction. Thus, the support function based on p_0 satisfies

$$f_{p_0}(p) = \langle p - p_0, N(p)\rangle \neq 0$$

for every $p \in \Sigma^n$.

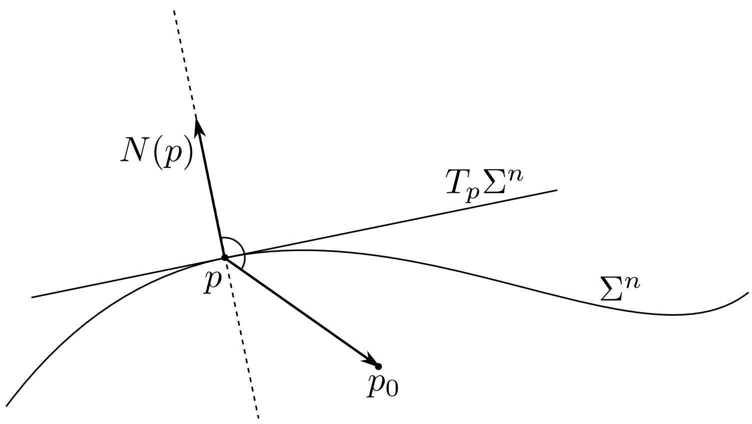

FIGURE 1. Support function based on p_0

We assume, without loss of generality, that $f_{p_0} > 0$ everywhere in Σ^n. The case when $f_{p_0} < 0$ is analogous. By using Newton's inequality

$$\frac{(\operatorname{trace} A)^2}{n} \leq |A|^2,$$

Equation (3.1), and the hypothesis $|A|^2 \leq 1/2$, we have

$$f^2 = 4H^2 = 4(\operatorname{trace} A)^2 \leq 4n|A|^2 \leq 2n. \tag{3.3}$$

This implies

$$-\sqrt{2n} \leq f \leq \sqrt{2n}$$

and, thus, there exist $m = \inf_{\Sigma^n} f$ and $d = \sup_{\Sigma^n} f$. If $m \leq 0$, then

$$\mathcal{L}(f-m) + \left(|A|^2 - \frac{1}{2}\right)(f-m) = -\left(|A|^2 - \frac{1}{2}\right) m \leq 0.$$

If $m = \min_{\Sigma^n} f$, i.e., if f reaches a minimum, then by the Hopf maximum principle (Lemma 2.1, item (ii)), applied to $f - m$, we can conclude that f is constant. On the other hand, if $m > 0$, then $d = \sup_{\Sigma^n} f > 0$. Thus if f reaches (positive) a maximum, i.e., $d = \max_{\Sigma^n} f$ then, applying the Hopf maximum principle (Lemma 2.1, item (ii)), to equation (3.2), we conclude that f is constant.

On the other hand, Dajczer and Tojeiro, see [8], Theorem 1, p.296, proved that the only hypersurfaces of $\mathbb{R}^{n+1}$ with constant support function f are the cylinders, spheres and hyperplanes. The conclusion that $\Sigma^n = \mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$, comes from Equation (3.1).

Thus we need to prove only that f reaches a minimum. The proof that f reaches a maximum is identical.

Let $\{p_k\}$ be a sequence of points in Σ^n such that $f(p_k) \to m$ when $k \to \infty$. For each p_k consider q_k the projection of p_0 over $T_{p_k}\Sigma^n$ (see Figure 2).

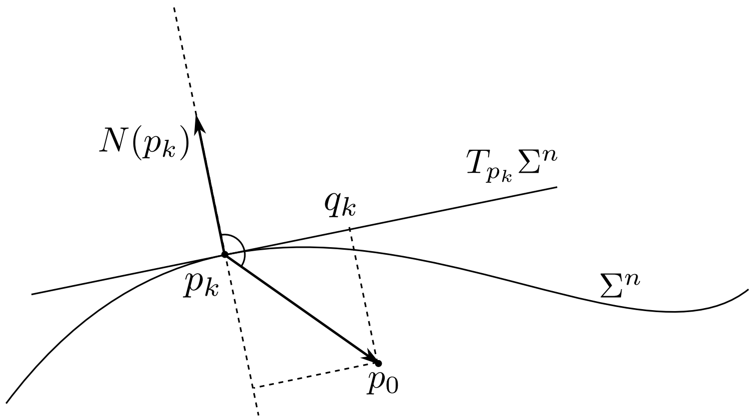

FIGURE 2. Projection of p_0 over $T_{p_k}\Sigma^n$

Since

$$\operatorname{dist}(q_k, p_0) = |q_k - p_0| = |\operatorname{proj}_{N(p_k)}(p_k - p_0)| = |\langle p_k - p_0, N(p_k)\rangle| = f_{p_0}(p_k),$$

where $\operatorname{proj}_u v$ denotes the projection of the vector v over the vector u, and

$$|f_{p_0}(q)| = |f(q) - \langle p_0, N(q)\rangle| \leq |f(q)| + |p_0| \leq \sqrt{2n} + |p_0|,$$

we have that $\{q_k\}$ is a bounded sequence in $\bigcup_{p\in\Sigma^n} T_p\Sigma^n = \mathbb{R}^{n+1} - W$. Moreover, since W is open, we have that $\mathbb{R}^{n+1} - W$ is closed. Thus, passing to a subsequence if necessary, we can deduce that q_k converges to a point $q_1 \in \mathbb{R}^{n+1} - W$. Let $p_1 \in \Sigma^n$ such that $q_1 \in T_{p_1}\Sigma^n$. This implies

$$f(p_1) = \lim_{k\to\infty} f(p_k) = m,$$

i.e, m is a minimum for f. $\square$

Proof of Theorem 1.2. Since the codimension is one and Σ^n is a self-shrinker, using Equation (2.22) of Proposition 2.1 for $\varepsilon = -1/2$, we obtain

$$\mathcal{L}f = \left(\frac{1}{2} - |A|^2\right) f.$$

Since $H \geq 0$ and $H = -\frac{1}{2}f$, we have that $f \leq 0$. Thus, if $|A|^2 \geq 1/2$, then $\mathcal{L}f \geq 0$. Since $f \leq 0$, there exists $d = \sup_{\Sigma^n} f$. Thus, if f reaches a maximum, i.e., $d = \max_{\Sigma^n} f$, then by using the Hopf maximum principle (Lemma 2.1, item (i)), we conclude that f is constant. Therefore, we need to prove only that f reaches a maximum.

Let $\{p_k\}$ be a sequence of points in Σ^n such that $f(p_k) \to d$ when $k \to \infty$. For each p_k consider q_k the projection of p_0 over $T_{p_k}\Sigma^n$. Since

$$\operatorname{dist}(q_k, 0) = |q_k| = |\operatorname{proj}_{N(p_k)}(p_k)| = |\langle p_k, N(p_k)\rangle| = f(p_k)$$

and $f(p_k)$ is a bounded sequence (since it converges), we have that $\{q_k\}$ is a bounded sequence in $\bigcup_{p\in\Sigma^n} T_p\Sigma^n = \mathbb{R}^{n+1} - W$. Moreover, since W is open, we have that $\mathbb{R}^{n+1} - W$ is closed. Thus, passing to a subsequence if necessary, we can deduce that q_k converges to a point $q_1 \in \mathbb{R}^{n+1} - W$. Let $p_1 \in \Sigma^n$ such that $q_1 \in T_{p_1}\Sigma^n$. This implies

$$f(p_1) = \lim_{k\to\infty} f(p_k) = d,$$

i.e, d is a maximum for f. □

Proof of Theorem 1.3. Since $\nabla^\perp \eta = 0$, where $\eta = \mathbf{H}/|\mathbf{H}|$, then $s_{1\beta} \equiv 0$, which implies that $A_\beta X^\top = 0$ for every $\beta = 2, \ldots, k$. Since $\operatorname{trace} A_\beta = 0$ and the dimension is two, we have that $A_\beta = 0$.

On the other hand, in $\mathbb{R}^{2+k}$ the position vector is a conformal vector with conformal factor $\varphi = 1$. In this case, since the codimension is $k \geq 2$ and Σ^2 is a self-shrinker, we have

$$f = \langle X, \eta\rangle = |X^\perp| = 2|\mathbf{H}| > 0.$$

Thus, by the Proposition 2.1, we have, for $c = -1/2$,

$$\mathcal{L}f = \left(\frac{1}{2} - |A|^2\right) f. \tag{3.4}$$

- **i)** If $|A|^2 \leq 1/2$, then $\mathcal{L}f \geq 0$. Since $f^2 \leq 4$ (see estimate (3.3) in the proof of Theorem 1.1), there exists $d = \sup_{\Sigma^2} f$. Thus, if f reaches a maximum, i.e., $d = \max_{\Sigma^2} f$, then by using the Hopf maximum principle (Lemma 2.1, item (i)), we conclude that f is constant.
- **ii)** If $|A|^2 \geq 1/2$ then $\mathcal{L}f \leq 0$. Since $f > 0$, there exists $m = \inf_{\Sigma^2} f$. Thus, if f reaches a minimum, i.e., $m = \min_{\Sigma^2} f$, then by using the Hopf maximum principle, (Lemma 2.1, item (i)), we conclude that f is constant.

The proof that f reaches a maximum or a minimum is identical to that presented in the proof of Theorem 1.2.

Thus, in both cases, f is constant, which implies that $|A|^2 = 1/2$. Since the second fundamental α satisfies

$$\begin{aligned}\mathcal{L}|\alpha|^2 = 2|\nabla\alpha|^2 + |\alpha|^2 - 2\sum_{\beta\neq\delta}|[A_\beta, A_\delta]|^2 \\ - 2\sum_{\beta,\delta}\left(\sum_{i,j=1}^{2}\langle\alpha(e_i,e_j),\eta_\beta\rangle\langle\alpha(e_i,e_j),\eta_\delta\rangle\right)^2 \\ = 2|\nabla\alpha|^2 + |\alpha|^2 - 2\sum_{\beta\neq\delta}|A_\beta\circ A_\delta - A_\delta\circ A_\beta|^2 \\ - 2\sum_{\beta,\delta}\left(\sum_{i,j=1}^{2}\langle A_\beta(e_i),e_j\rangle\langle A_\delta(e_i),e_j\rangle\right)^2 \\ = 2|\nabla\alpha|^2 + |\alpha|^2 - 2\sum_{\beta\neq\delta}|A_\beta\circ A_\delta - A_\delta\circ A_\beta|^2 \\ - 2\sum_{\beta,\delta}(\mathrm{trace}(A_\beta\circ A_\delta))^2\end{aligned}$$

(see [9], p.5069, Eq. (2.5)) and $A_\beta = 0$, $\beta = 2,\ldots,k$, we have $|\alpha| = |A|$ and

$$\mathcal{L}|A|^2 = 2|\nabla A|^2 + |A|^2 - 2|A|^4.$$

Thus $|A|^2 = 1/2$ implies that $|\nabla A|^2 = 0$. Therefore Σ^2 is isoparametric and thus $\Sigma^2 = \mathbb{S}^1(\sqrt{2})\times\mathbb{R}$ or $\Sigma^2 = \mathbb{S}^2(2)$. $\square$

Proof of Theorem 1.4. If Σ^n is a self-expander such that $f = 2|\mathbf{H}| > 0$, and $\mathbf{H}/|\mathbf{H}|$ is parallel, then, by Proposition 2.1,

$$\mathcal{L}f + \left(|A|^2 + \frac{1}{2}\right)f = 0.$$

If the codimension is one, then $f = \langle X, N\rangle = 2H$. Since $H \neq 0$, we can choose an orientation such that $H > 0$, i.e, $f > 0$. Thus $\mathcal{L}f \leq 0$. Since f is bounded below, there exists $m = \inf_{\Sigma^n} f$. Since W is open, reasoning as in the proof of Theorem 1.2, we can prove that m is actually a minimum. Therefore, by the Hopf maximum principle, we can see that f is constant, which implies

$$\left(|A|^2 + \frac{1}{2}\right)f = 0,$$

but it is impossible, since $f > 0$. $\square$

4. Example of a non proper surface with nonempty W

In this section we show that the condition $W \neq 0$ does not implies, necessarily, that the immersion is proper or has polynomial volume growth. This implies that the assumption that $W \neq \emptyset$ gives a new class of submanifolds, distinct from the proper ones, and distinct from those with polynomial volume growth. It will be done exhibiting the following class of examples of non proper immersions with nonempty W and volume growth larger than polynomial.

Let $\Sigma^2 = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$ be a cylinder over a curve Γ parameterized by

$$\Gamma(t) = (1 + b(t))(\cos t, \sin t), \tag{4.1}$$

where $b(t)$ is a smooth function satisfying the following conditions:

(i) $\lim_{t\to\infty} b(t) = 0$;
(ii) $\lim_{t\to-\infty} b(t) = m,\ m \in (0, \infty]$.
(iii) $b'(t) < 0$;
(iv) $b(t) \geq |b'(t)|$;
(v) $|b'(t)| < 1$.

An example of a class of functions $b(t)$ which satisfies the conditions (i) to (v) is

$$b(t) = \frac{m}{\pi}\left(\frac{\pi}{2} - \arctan(at)\right),\ m > 0,\ 0 < a \leq 1. \tag{4.2}$$

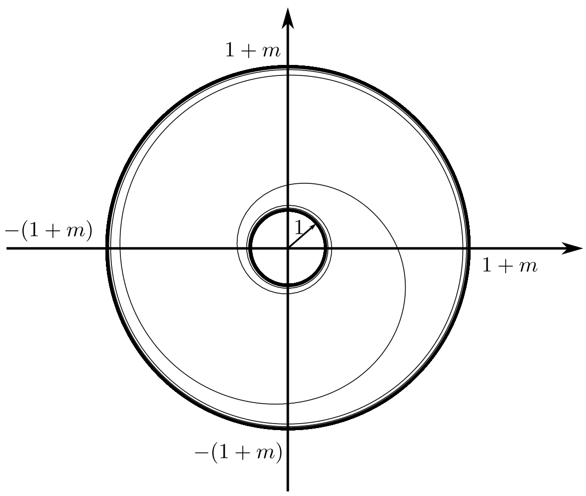

Figure 3. Trace of $\Gamma(t) = \left(1 + \frac{m}{\pi}\left(\frac{\pi}{2} - \arctan(at)\right)\right)(\cos t, \sin t),\ m > 0,$ $0 < a \leq 1$

The curve Γ is a spiral asymptotic to the unit circle, giving an infinite number of turns around this circle. If $m < \infty$, then Γ is also asymptotic to the circle with center at the origin and radius $1 + m$ (see Fig. 3), also giving an infinite number of turns around this circle. This implies that $\Sigma^2 = \Gamma \times \mathbb{R}$ is asymptotic to the cylinder $\mathbb{S}^2 \times \mathbb{R}$ (and also to $\mathbb{S}^2(1+m) \times \mathbb{R}$ in the case when $m \neq \infty$), giving an infinite number of turns inside any compact set which contains a slice of the cylinder $\mathbb{S}^2(1+\varepsilon) \times \mathbb{R}$, $\varepsilon > 0$. Therefore, it is a non proper immersion.

We shall prove $\Sigma = \Gamma \times \mathbb{R}$ satisfies

$$W := \mathbb{R}^3 \backslash \bigcup_{p \in \Sigma} T_p\Sigma = \mathbb{D}^2 \times \mathbb{R}, \tag{4.3}$$

where $\mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$.

In fact, notice that, since all the tangent planes of the cylinder $\Sigma^2 = \Gamma \times \mathbb{R}$ are vertical planes over the tangent lines of Γ, if we show that every point of every tangent line of Γ has a distance ≥ 1 of the origin, we will obtain that $W = \mathbb{D}^2 \times \mathbb{R}$. The tangent line to Γ at t is

$$R_t(s) = \Gamma(t) + s\Gamma'(t),$$

which implies

$$|R_t(s)|^2 = |\Gamma(t)|^2 + 2s\langle \Gamma(t), \Gamma'(t)\rangle + s^2|\Gamma'(t)|^2.$$

Since

$$\Gamma'(t) = b'(t)(\cos t, \sin t) + (1 + b(t))(-\sin t, \cos t),$$

we have

$$\begin{aligned} |R_t(s)|^2 &= (1+b(t))^2 + 2s(1+b(t))b'(t) + s^2(b'(t))^2 + s^2(1+b(t))^2 \\ &= [1 + b(t) + sb'(t)]^2 + s^2(1+b(t))^2. \end{aligned}$$

By using condition (iii), and defining $c(t) := 1 + b(t) - |b'(t)|$ (notice that $c(t) \geq 1$ by the condition (iv)), we have

$$\begin{aligned} |R_t(s)|^2 &= [1 + b(t) - s|b'(t)|]^2 + s^2(1+b(t))^2 \\ &= [c(t) + (1-s)|b'(t)|]^2 + s^2(1+b(t))^2 \\ &= (c(t))^2 + s^2 + 2[(1-s)c(t)|b'(t)| + s^2b(t)] + (1-s)^2|b'(t)|^2 + s^2(b(t))^2 \\ &\geq (c(t))^2 + s^2 + 2[(1-s)c(t)|b'(t)| + s^2b(t)] + [(1-s)^2 + s^2]|b'(t)|^2 \\ &> (c(t))^2 + s^2 + 2[(1-s)c(t)|b'(t)| + s^2b(t)]. \end{aligned}$$

On the other hand, by using properties (iv) and (v),

$$\begin{aligned}(1-s)c(t)|b'(t)| + s^2 b(t) &= (1-s)c(t)|b'(t)| + s^2 c(t)|b'(t)| - s^2 c(t)|b'(t)| + s^2 b(t) \\ &= (1-s+s^2)c(t)|b'(t)| + s^2[b(t) - c(t)(1+b(t)-c(t))] \\ &= (3/4 + (s-1/2)^2)c(t)|b'(t)| + s^2(c(t)-1)(c(t)-b(t)) \\ &= (3/4 + (s-1/2)^2)c(t)|b'(t)| + s^2(b(t)-|b'(t)|)(1-|b'(t)|) \\ &\geq 0.\end{aligned}$$

Thus

$$|R_t(s)|^2 > c^2 + s^2 \geq 1.$$

This will imply, by the previous discussion, that (4.3) holds.

On the other hand, the curvature $k(t)$ of $\Gamma(t)$ is

$$k(t) = \frac{(1+b(t))^2 + 2(b'(t))^2 - (1+b(t))b''(t)}{[(1+b(t))^2 + (b'(t))^2]^{3/2}}.$$

Moreover, if $m \neq \infty$, by using that $b(t)$ is monotone, $\lim_{t\to\infty} b(t) = 0$ and $\lim_{t\to-\infty} b(t) = m$, we have that

$$\lim_{t\to\pm\infty} b'(t) = \lim_{t\to\pm\infty} b''(t) = 0.$$

Thus,

$$\lim_{t\to\infty} k(t) = 1 \text{ and } \lim_{t\to-\infty} k(t) = \frac{1}{1+m}.$$

Therefore, the mean curvature of $\Sigma = \Gamma \times \mathbb{R}$ (which is equal to the curvature of Γ) is bounded. Since, in the cylinder, the normal is horizontal (i.e., perpendicular to the z-axis) and Γ is contained in the ring limited by the circles of center at the origin and radius 1 and $1+m$ (see Fig. 3), we deduce that the support function $\langle X, N\rangle$ is bounded. This will imply that the Gaussian weighted mean curvature

$$H_f = H + \frac{1}{2}\langle X, N\rangle,$$

for the Gaussian weight $|X|^2/4$, is also bounded. Applying Theorem 1.4, p. 6 of [4], we conclude also that Σ does not have polynomial volume growth. Thus we obtain a class of non proper surfaces, which does not have polynomial volume growth, and such that $W \neq \emptyset$.

Remark 4.1. By choosing $\Gamma(t) = (d + b(t))(\cos t, \sin t)$, $d > 0$, the curvature $k(t)$ of $\Gamma(t)$ satisfies

$$\frac{1}{d+m} < k(t) < \frac{1}{d}.$$

Therefore, by choosing suitable values of $d > 0$ and $m > 0$, we obtain that $|A|^2$ can lie in any open interval of $(0, \infty)$, including the interval $[0, 1/2]$ of Theorem 1.1 and the interval $[1/2, \infty)$ of Theorem 1.2.

Remark 4.2. Another example, simpler then the previous one, but with unbounded support function $\langle X, N\rangle$ is given by taking $b(t) = e^{-t}$ in (4.1). In this case we obtain

$$\begin{aligned}
|R_t(s)|^2 &= (1 + (1-s)e^{-t})^2 + (1 + e^{-t})^2 s^2 \\
&= 1 + s^2 + 2(1 - s + s^2)e^{-t} + [(1-s)^2 + s^2]e^{-2t} \\
&= 1 + s^2 + 2(1 - s + s^2)e^{-t} + [(1-s)^2 + s^2]e^{-2t} \\
&= 1 + s^2 + 2[3/4 + (s - 1/2)^2]e^{-t} + [(1-s)^2 + s^2]e^{-2t} \\
&> 1 + s^2 \geq 1.
\end{aligned}$$

This will imply, by the previous discussion, that $W := \mathbb{R}^3 \backslash \bigcup_{p\in\Sigma} T_p\Sigma = \mathbb{D}^2 \times \mathbb{R}$. On the other hand, since $\lim_{t\to\infty} e^{-t} = 0$, then $\Gamma(t) = (1 + e^{-t})(\cos t, \sin t)$ is a spiral asymptotic to the unit circle. This gives that $\Sigma^2 = \Gamma \times \mathbb{R}$ is asymptotic to the cylinder, giving an infinite number of turns inside any compact set which contains a slice of the cylinder $\mathbb{S}^2(1 + \varepsilon) \times \mathbb{R}$, $\varepsilon > 0$. Therefore, it is a non proper immersion with $W \neq 0$.

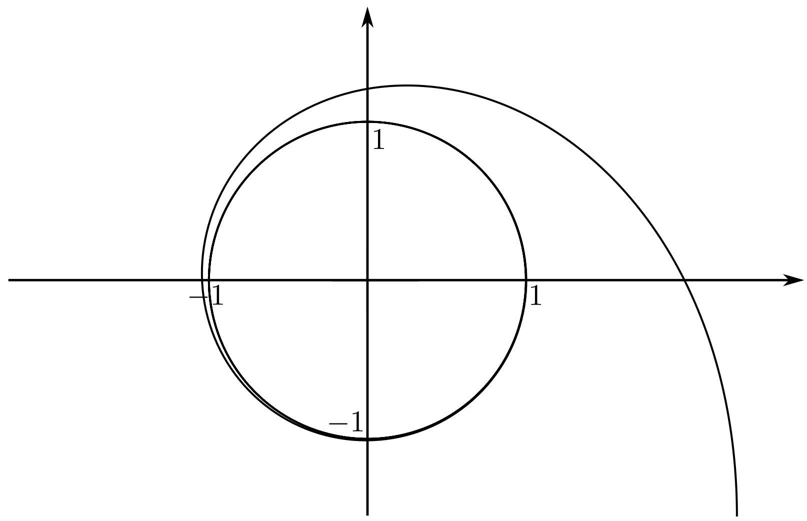

FIGURE 4. Trace of $\Gamma(t) = (1 + e^{-t})(\cos t, \sin t)$

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