HYPERSURFACES WHOSE TANGENT GEODESICS OMIT A NONEMPTY SET

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Dedicated to Manfredo P. do Carmo on his sixtieth birthday

1. Introduction

Let Q_c^{n+1} be an $n+1$-dimensional, simply-connected, complete Riemannian manifold with constant sectional curvature c. Let M^n be an n-dimensional connected manifold and $x : M^n \to Q_c^{n+1}$ be an immersion. For every point $p \in M^n$, let $(Q_c^n)_p$ be the totally geodesic hypersurface of Q_c^{n+1} tangent to $x(M^n)$ at $x(p)$.

We will denote by

$$W = Q_c^{n+1} | \bigcup_{p \in M} (Q_c^n)_p$$

the set of points which are omitted by the totally geodesic hypersurfaces tangent to $x(M^n)$. In this work we study the immersions for which the set W is nonempty.

The first known result in this direction is due to Halpern. He proved in [6] that every compact hypersurface immersed in the euclidean space with nonempty W is diffeomorphic to the sphere and it is, in fact, embedded. We show that the same happens when the ambient space is Q_c^{n+1}, c arbitrary (see Proposition 4.1). If, in addition, the immersion is isometric with constant mean curvature, we prove that $x(M^n)$ is, actually, a geodesic sphere (see Theorem 4.2). The case when x is minimal, was proved by Pogorelov in [11].

Halpern also proved in [6] that if M^n is compact and $x : M^n \to \mathbb{R}^{n+1}$ is an immersion with nonempty W, then W is, in fact, open. In the case M^n is complete noncompact there are several examples where the set W is nonempty, but not open. One such example is the hyperboloid of one sheet in $\mathbb{R}^3$, for which the set W consists of a single point. However, Hasanis and Koutroufiots proved in [7] that if an immersion $x : M^2 \to Q_c^3$, $c \geq 0$, is minimal with nonempty W, then x is totally geodesic. In particular, W is open. The proof of this result uses strongly the hypothesis that M has dimension two. We show that the same holds for arbitrary dimensions if we assume, in addition, that the set W is open (Theorem 3.1). Recently, the first author, in his Doctoral thesis at IMPA, gave examples of nontotally geodesics minimal hypersurfaces in $\mathbb{R}^{2n}$, $n \geq 4$, with nonempty W.

This paper is organized as follows. In section 2, we extend for Q_c^{n+1}, the notions of position vector and support function. This is essentially known (see, for instance, Heintze [8]) but, since we need the details, we will present a full exposition. A geometric interpretation of the support function is also presented in this section. In section 3, we study minimal immersions with nonempty W. We prove Theorem (3.1) above and show that every minimal hypersurface in Q_c^{n+1}, $c \leq 0$, with nonempty W is stable. In section 4, we study the compact hypersurfaces in Q_c^{n+1} with nonempty W.

We would like to thank M. P. do Carmo for suggesting this topic to us and for some ideas that lead us to Theorem (3.1).

2. Support Function in Spaces of Constant Curvature

Let M^n be an oriented Riemannian manifold, and let $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. Given $p_0 \in \mathbb{R}^{n+1}$, let $X(p) = p - p_0$ be the position vector with origin p_0. The support function $g : M^n \to \mathbb{R}$ of the immersion x is given by

$$g(p) = \langle x(p), N(p) \rangle,$$

where N is a unit normal vector field of x. We will extend, for Q_c^{n+1}, $c \neq 0$, the notions of position vector and support function.

Let S_c be a solution of the equation $y'' + cy = 0$, with initial conditions $y(0) = 0$ and $y'(0) = 1$. Then

$$S_c(r) = \begin{cases} r, & \text{if} \quad c = 0, \\ \sin(\sqrt{c}\, r)/\sqrt{c}, & \text{if} \quad c > 0, \\ \sinh(\sqrt{-c}\, r)/\sqrt{-c}, & \text{if} \quad c < 0. \end{cases}$$

For every point $p_0 \in Q_c^{n+1}$, we will consider the function $r(\cdot) = d(\cdot, p_0)$, where d is the distance function of Q_c^{n+1}, and we will denote by $\operatorname{grad} r$ the gradient of the function r in Q_c^{n+1}. We know that when $c = 0$ the position vector with origin

p_0 is given by $X(p) = S_0(r)$ grad r. By analogy, the vector field, in Q_c^{n+1}, $X(p) = S_c(r)$ grad r will be called *position vector* with origin p_0. When $c > 0$, the distance function is differentiable in $Q_c^{n+1}/\{p_0, -p_0\}$. Therefore, in this case, the position vector with origin p_0 is differentiable in $Q_c^{n+1}|\{p_0, -p_0\}$.

Let M^n be an oriented Riemannian manifold, $x : M^n \to Q_c^{n+1}$ an isometric immersion, and N an unit normal vector field of x. As in the case $c = 0$, the function $g : M^n \to \mathbb{R}$ defined by $g = \langle X, N\rangle$, where X is the position vector with origin p_0, will be called the *support function* of the immersion x. In the case $c > 0$, this function is differentiable if $x(M^n) \subseteq Q_c^{n+1}|\{p_0, -p_0\}$.

For the case $c = 0$, $|g(p)|$, $p \in M^n$, is the distance from p_0 to the tangent hyperplane to $x(M^n)$ at $x(p)$. We will now give a geometric interpretation of the support function g for $c \neq 0$ that generalizes the above.

In the case $c > 0$, we will assume that Q_c^{n+1} is the sphere of radius $1/\sqrt{c}$ in $\mathbb{R}^{n+2}$. Then, $|g(p)|$, $p \in M^n$, is the euclidean distance from the point p_0 to the hyperplane which contains the totally geodesic hypersurface tangent to $x(M^n)$ at $x(p)$. In fact, since

$$p_0 = \cos(\sqrt{c}\, r(p))\, p - \frac{\sin(\sqrt{c}\, r(p))}{\sqrt{c}} \text{ grad } r(p), \tag{1}$$

we have

$$\langle p_0, N(p)\rangle = -\frac{\sin(\sqrt{c}\, r(p))}{\sqrt{c}} \langle \text{grad } r(p),\, N(p)\rangle = -g(p). \tag{2}$$

So $|g(p)| = |\langle p_0, N(p)\rangle|$.

In the case $c < 0$, let L^{n+2} be the euclidean space $\mathbb{R}^{n+2}$ endowed with the Riemannian pseudo-metric $\langle\langle\ \rangle\rangle$, defined by

$$\langle\langle v, w\rangle\rangle = v_1 w_1 + v_2 w_2 + ... + v_{n+1} w_{n+1} - v_{n+2} w_{n+2},$$

where $v = (v_1, ..., v_{n+2})$ and $w = (w_1, ..., w_{n+2})$ are vectors in $\mathbb{R}^{n+2}$. Let $\mathbb{H}^{n+1}(c)$ be the hypersurface of L^{n+2} given by

$$\mathbb{H}^{n+1}(c) = \left\{ v \in L^{n+2};\ v_{n+2} > 0 \text{ and } \langle\langle v, v\rangle\rangle = \frac{1}{c} \right\}.$$

It is well known that $\mathbb{H}^{n+1}(c)$ with the induced metric is a model of the hyperbolic space Q_c^{n+1}, called *hyperboloid model*.

We can assume, without loss of generality, that $p_0 = (0, \cdots, 0, 1/\sqrt{-c})$. In this case, the euclidean distance from p_0 to the hyperplane that passes through the origin of $\mathbb{R}^{n+2}$ and contains the totally geodesic hypersurface, $(Q_c^n)_p$, tangent to $x(M^n)$ at $x(p)$, $p \in M^n$, is given by

$$\frac{|g(p)|}{\sqrt{1+2g(p)^2}}.$$

In fact, since

$$p_0 = \cosh(\sqrt{-c}\, r(p))p - \frac{\sinh(\sqrt{-c}\ r(p))}{\sqrt{-c}} \text{ grad } r(p), \tag{3}$$

we have that

$$\langle\langle p_0, N(p)\rangle\rangle = -g(p). \tag{4}$$

Let $N(p) = (N_1, ..., N_{n+1}, N_{n+2})$. Then $\langle\langle p_0, N(p)\rangle\rangle = -N_{n+2}$ and $\langle p_0, N(p)\rangle = N_{n+2}$, where $\langle\ ,\ \rangle$ is the usual inner product. Since $\langle\langle n(p), N(p)\rangle\rangle = 1$, $\langle\langle p, N(p)\rangle\rangle = 0$ and $\langle\langle v, N(p)\rangle\rangle = 0$ for every $v \in T_p(Q_c^n)_p$, we have that

$$\overline{N}(p) = \frac{(N_1, ..., N_{n+1}, N_{n+2})}{\sqrt{1+2N_{n+2}^2}}.$$

is an unit vector in $\mathbb{R}^{n+2}$ orthogonal to the hyperplane that passes through the origin of $\mathbb{R}^{n+2}$ and contains $(Q_c^n)_p$. Therefore, the euclidean distance from p_0 to this hyperplane is given by

$$|\langle p_0, \overline{N}(p)\rangle| = \left|\frac{-N_{n+2}}{\sqrt{1+2N_{n+2}^2}}\right| = \frac{|g(p)|}{\sqrt{1+2g(p)^2}}, \tag{5}$$

and this proves our assertion.

We now assume that the immersion $x : M^n \to Q_c^{n+1}$ has constant mean curvature H. Setting $\theta_c = S_c'$, we have in the case $c = 0$ that $\Delta g = -\|B\|^2 g - n\ H\ \theta_c$. The proposition below says that this equation holds for any c.

2.1 Proposition. *Let M^n be an oriented Riemannian manifold and let $x : M^n \to Q_c^{n+1}$ be an isometric immersion with constant mean curvature H. Then*

$$\Delta g = -\|B\|^2 g - n\ H\ \theta_c,$$

where Δ is the Laplacian in M^n and $\|B\|$ is the norm of the second fundamental form B of the immersion x.

Proof. The result was proved in [1] for the case $c = 0$. If $c > 0$ or $c < 0$, we have by Lemma (3.3) in [2] that

$$\Delta f = -\|B\|^2 \ f + n \ c \ H \ h,$$

where $f(p) = \langle N(p), p_0 \rangle$, $h(p) = \langle p, p_0 \rangle$ and $\langle \ , \ \rangle$ denotes the euclidean and Lorentz inner product, respectively. But, from (2) and (4), $g = -f$, and from (1) and (3), $\theta_c = ch$. Therefore

$$\Delta g = -\|B\|^2 \ g - n \ H \ \theta_c.$$

The mean value equality (6) below generalizes for $c \neq 0$ the Minkowski's equality in $\mathbb{R}^{n+1}$ $(c = 0, \ \theta_c = 1)$. For completeness, we will present a complete proof.

2.2 Proposition. (**Heintze** [8]**, pag. 19**). *Let M^n be a compact Riemannian manifold and let $x : M^n \to Q_c^{n+1}$ be an isometric immersion. Then*

$$\int_M H \ g \ dA = -\int_M \theta_c \ dA, \tag{6}$$

where H is the mean curvature of x.

Proof. Let X be the position vector with origin p_0 and $e_1, ..., e_n$ be a local orthonormal frame of TM. Denote by div_M the divergence in M^n, and by X^t and X^N the tangent and normal components, respectively, of the vector X.

Since $\langle X^N, e_i \rangle = 0$, we have that $\langle \overline{\nabla}_{e_i} X^N, e_i \rangle = -\langle X, (\overline{\nabla}_{e_i} e_i)^N \rangle$, and so

$$\begin{aligned} \operatorname{div}_M \ X^T \quad &= \quad \sum_{j=1}^n \langle \overline{\nabla}_{e_j} X^T, e_j \rangle = \sum_{j=1}^n \langle \overline{\nabla}_{e_j} X, e_j \rangle - \sum_{j=1}^n \langle \overline{\nabla}_{e_j} X^N, e_j \rangle \\ &= \quad \sum_{j=1}^n \langle \overline{\nabla}_{e_j} X, e_j \rangle + \sum_{j=1}^n \langle X, (\overline{\nabla}_{e_j} e_j)^N \rangle, \end{aligned}$$

where $\overline{\nabla}$ is the Riemannian connection of Q_c^{n+1}.

On the other hand, we have that $\sum_{j=1}^n \langle \overline{\nabla}_{e_j} X, e_j \rangle = n \ \theta_c$. In fact,

$$\begin{aligned} \sum_{j=1}^n \langle \overline{\nabla}_{e_j} X, e_j \rangle &= \sum_{j=1}^n \langle \overline{\nabla}_{e_j} (S_c(r) \operatorname{grad} \ r), e_j \rangle \\ &= \theta_c(r) \sum_{j=1}^n \langle \operatorname{grad} \ r, e_j \rangle^2 + S_c(r) \sum_{j=1}^n \langle \overline{\nabla}_{e_j} \operatorname{grad} \ r, e_j \rangle. \end{aligned} \tag{7}$$

But, as we can see in (Jorge, Koutroufiotis [10], pg. 713), we have that

$$\langle \overline{\nabla}_v \,\text{grad}\; r, w\rangle = \frac{\theta_c}{S_c}(\langle v, w\rangle - \langle \text{grad}\; r, v\rangle\langle \text{grad}\; r, w\rangle), \tag{8}$$

for any vector fields v, w in Q_c^{n+1}.

Then, from (7) and (8),

$$\begin{aligned}\sum_{j=1}^{n}\langle \overline{\nabla}_{e_j} X, e_j\rangle &= \theta_c(r)\sum_{j=1}^{n}\langle \text{grad}\; r, e_j\rangle^2 + \theta_c(r)\sum_{j=1}^{n}(1 - \langle \text{grad}\; r, e_j\rangle^2) \\ &= n\;\theta_c.\end{aligned}$$

Thus, since $\sum_{j=1}^{n}(\overline{\nabla}_{e_j} e_j)^N = H\;N$,

$$\text{div}_M\; X^T = n\;\theta_c + n\;H\;g.$$

By integrating the above expression over M^n, we obtain

$$\int_M H\;g\;dA = -\int_M \theta_c\;dA.$$

This complete the proof.

2.3 Remark. In [8], assuming only that the sectional curvature of the ambient space is bounded above, it is proven that an inequality still holds in the last proposition.

3. Minimal Hypersurfaces with Nonempty W

3.1 Theorem. *Let M^n be a complete Riemannian manifold and let $x : M^n \to Q_c^{n+1}$ be an isometric minimal immersion. If the set W is open and nonempty, then x is totally geodesic.*

Proof. Let $p_0 \in W$ and X be the position vector with origin p_0. For each point $p \in M^n$, let $N(p)$ be the unit normal vector to $X(M^n)$ at $x(p)$ such that $\langle X(p), N(p)\rangle > 0$. This gives M^n an orientation, according to which the support function $g = \langle X, N\rangle$ is positive.

Let $d = \inf\{g(p);\; p \in M^n\}$. Assume that there is a point $p \in M^n$ such that $g(p) = d$. Since, from (1.2), $\Delta g = -\|B\|^2 g$, we have $\Delta g \leq 0$. Then, from the Maximum Principle, g is constant equal to d. Thus $\|B\| \equiv 0$, i.e., x is totally geodesic, for $\Delta g = 0$ and g vanishes nowhere.

Therefore, the proof will be complete if we show that there is a point $p \in M^n$ such that $g(p) = d$. For that, we will consider a sequence of points $\{p_k\}_{k\geq 0}$ in M^n such that $g(p_k) \to d$, when $k \to \infty$.

We will treat separately the cases $c = 0$, $c > 0$ and $c < 0$, and we will assume, without loss of generality, that $c = 1$, when $c > 0$ and $c = -1$ when $c < 0$.

Case $c = 0$**.** For each point p_k, we will consider the point q_k, intersection of $T_{p_k}M^n$ with the perpendicular line to $T_{p_k}M^n$ which passes through p_0. Since $d(q_k, p_0) = g(p_k)$ is a bounded sequence, there is a subsequence $\{q_{k_j}\}$ that converges to a point $q \in \mathbb{R}^{n+1}$. Then $q \in T_pM^n$ for some point $p \in M^n$, since $\cup_{p\in M} T_pM^n$ is closed and $q_k \in T_{p_k}M^n$ for every k. Therefore $g(p) = d(p_0,\ T_pM) = d$, for $d(p_0, q) = d$ and

$$d \leq d(p_0, T_pM) \leq d(p_0, q) = d.$$

Case $c = 1$**.** For each point p_k, let s_k be the orthogonal projection of p_0 over the hyperplane of $\mathbb{R}^{n+2}$ which contains $(Q_c^n)_{p_k}$ and let q_k be the intersection of $(Q_c^n)_{p_k}$ with the line which passes through the origin and the point s_k. Since, for every k, $q_k \in Q_c^{n+1} = S^{n+1}$ and $s_k \in \mathbb{B}^{n+2} = \{p \in \mathbb{R}^{n+2};\ \|p\| \leq 1\}$, there is a subsequence k_j such that $\{q_{k_j}\}$ converges to a point $q \in S^{n+1}$ and $\{s_{k_j}\}$ converges to a point $s \in \mathbb{B}^{n+2}$. Then $q \in (Q_c^n)_p$ for some point $p \in M^n$, since $\cup_{p\in M}(Q_c^n)_p$ is closed in S^{n+1}. Moreover, s and q are colinear, because s_k and q_k are colinear for every k. Thus s belongs to the hyperplane L_p of $\mathbb{R}^{n+2}$ that contains $(Q_c^n)_p$. Since $g(p_k) = d(s_k, p_0)$ and

$$d \leq g(p) = d(p_0, L_p) \leq d(p_0, s) = \lim_{k\to\infty} d(s_k, p_0) = d,$$

we have that $g(p) = d$.

Case $c = -1$**.** To prove the theorem in this case we will use the hyperboloid model of Q_c^{n+1} (cf. section 2). In the same way as in the preceeding case, we can define the point s_k. Form (5), the euclidean distance of p_0 to the hyperplane of $\mathbb{R}^{n+2}$ which passes through the origin and contains $(Q_c^n)_{p_k}$ is given by

$$\|s_k - p_o\| \frac{g(p_k)}{\sqrt{1 + 2g(p_k)^2}},$$

where $\|\ \ \|$ is the euclidean norm.

We assert that $\langle\langle s_k, s_k\rangle\rangle < 0$, where $\langle\langle\ ,\ \rangle\rangle$ is the Lorentz inner product. If $\langle\langle s_k, s_k\rangle\rangle \geq 0$, we have

$$\|s_k - p_o\| \geq \frac{\sqrt{2}}{2},$$

since s_k and $s_k - p_0$ are perpendicular. Then

$$\frac{g(p_k)^2}{1+2g(p_k)^2} \geq \frac{1}{2},$$

which is a contradiction and proves the assertion.

Let $\lambda_k > 0$ be such that $\lambda_k^2\langle\langle s_k, s_k\rangle\rangle = -1$, and let $q_k = \lambda_k s_k$, i.e., q_k is the intersection of $(Q_c^n)_{p_k}$ with the line which passes through the origin an through s_k.

Since the sequence $\{s_k\}_{k\geq 0}$ is bounded, by passing to a subsequence if necessary, there exists a point s such that $s_k \to s$, as $k \to \infty$. We can prove, as before, that $\langle\langle s, s\rangle\rangle < 0$, since s and $s - p_0$ are perpendicular and $\|s - p_0\|^2 = \dfrac{d^2}{1+2d^2}$. Thus the sequence $\{q_k\}$ is bounded, since the sequence $\left\{\dfrac{1}{\langle\langle s_k, s_k\rangle\rangle}\right\}$ is bounded from below by a positive constant and

$$\|q_k\|^2 = -\frac{\|s_k\|^2}{\langle\langle s_k, s_k\rangle\rangle}.$$

Let $\{q_{k_j}\}$ be a subsequence which converges to a point $q \in Q_c^{n+1}$. Since $\cup_{p\in M}(Q_c^n)_p$ is closed, and $q_k \in (Q_c^n)_{p_k}$ for every k, we have that $q \in (Q_c^n)_p$, for some point $p \in M^n$. Moreover, s belongs to the hyperplane L_p of $\mathbb{R}^{n+2}$ which contains $(Q_c^n)_p$, for s and q are colinear. Thus $g(p) = d$, since

$$\frac{g(p)}{\sqrt{1+2g(p)^2}} = d(p_0, L_p) \leq ||s - p_0|| = \frac{d}{\sqrt{1+2d^2}},$$

where $d(p_0, L_p)$ is the euclidean distance from p_0 to L_p.

When the set W is only nonempty, we have obtained the following result, for the cases $c \leq 0$.

3.2 Proposition. *Let M^n be a complete Riemannian manifold and $x : M^n \to Q_c^{n+1}$, $c \leq 0$, be a minimal isometric immersion. If W is nonempty, then x is stable.*

Proof. Let $p_0 \in W$ and X be the position vector with origin p_0. Since $p_0 \in W$, we can choose an orientation N in M^n for which the support function $g = \langle X, N\rangle$

is positive. From Proposition (2.1), $\Delta g + \|B\|^2\ g = 0$. In ([4], Theorem 1) F. Colbrie and R. Schoen proved that an operator of the type $\Delta + q$, where $q : M \to \mathbb{R}$ is a differentiable function, is positive if and only if there is a positive differentiable function $f\colon M \to \mathbb{R}$ such that $\Delta f + qf = 0$. Since the support function is positive and $\Delta g + \|B\|^2\ g = 0$, the operator $\Delta + \|B\|^2$ is positive definite, i.e.,

$$\int_M (|\operatorname{grad}\ f|^2 - \|B\|^2\ f^2)\ dA\ >\ 0,$$

for every nonzero function $f\colon M \to \mathbb{R}$ with compact support in M^n. Then, if $c \leq 0$,

$$\int_M (|\operatorname{grad}\ f|^2 - (\|B\|^2 + nc)f^2)\ dA\ >\ 0$$

for every such function f. Since the Ricci curvature of Q_c^{n+1} is nc, we obtain that the operator $A + \|B\|^2 + \operatorname{Ricc}\ (N)$ is positive definite, i.e., x is stable.

3.3 Remark. From the result of Hasanis and Koutroufiotis mentioned in the introduction we have that the above proposition doesn't hold for $c > 0$.

3.4 Remark. In ([5], pg. 57) J. Gomes gave examples of stable minimal hypersurface in Q_c^{n+1}, $c < 0$, which are not totally geodesic. For these hypersurfaces it is easy to see that the set W is empty. Then the converse of the above proposition is not true.

4. Compact Hypersurfaces with Nonempty W

We first generalize the result of Halpern mentioned in the introduction.

4.1 Proposition. *Let M^n be a connected, compact manifold, and let $x : M^n \to Q_c^{n+1}$ be an immersion. If W is nonempty, then M^n is diffeomorphic to the sphere S^n and x is an embedding.*

Proof. The case $c = 0$ has been proved by Halpern in [6]. Let $c < 0$ and let $\mathbb{R}_+^{n+1} = \{(v_1, ..., v_{n+1}) \in \mathbb{R}^{n+1};\ v_{n+1} > 0\}$ be the euclidean half space with the usual metric. To prove the proposition in this case, it is sufficient to notice that there is a diffeomorphism $B : Q_c^{n+1} \to \mathbb{R}_+^{n+1}$ which preserves the totally geodesic submanifolds. This mapping, usually known as Beltrami's mapping can be found, for instance in (do Carmo, Warner [3], pg. 142). Since a diffeomorphism preserves tangency, the above result follows immediately from the case $c = 0$.

To prove the case $c > 0$, we will make use of an argument which was used by Halpern in [6]. Let $p_0 \in W$. We can assume, without loss of generality, that $c = 1$ and $p_0 = e_{n+1} = (0, ..., 0, 1)$. Since $e_{n+1} \in W$, $-e_{n+1} \notin x(M^n)$, because every totally geodesic hypersurface of S^{n+1} which passes through $-e_{n+1}$ also passes through e_{n+1}. Since $x(M^n) \subseteq S^{n+1}|\{e_{n+1}, -e_{n+1}\}$, the function $r(\cdot) = d(\cdot, e_{n+1})$ is differentiable in $x(M^n)$, and $(\exp_{e_{n+1}})^{-1} : x(M^n) \to B_\pi(0)$ is well defined; here $\exp_{e_{n+1}}$ is the exponential map of S^{n+1} at e_{n+1} and

$$B_\pi(0) = \{v \in T_{e_{n+1}} S^{n+1};\ \|v\| < \pi\}.$$

Let $p \in S^{n+1}|\{e_{n+1}, -e_{n+1}\}$ and let $\gamma_p : [0, \pi) \to S^{n+1}$ be the geodesic such that $\gamma_p(0) = e_{n+1}$ and $\gamma_p(r(p)) = p$. There exists a unit vector $v(p) \in T_{e_{n+1}} S^{n+1}$, $v(p) = \exp^{-1}_{e_{n+1}}(p)|r(p)$, such that

$$\gamma_p(t) = \cos t\ e_{n+1} + \sin t\ v(p).$$

Since $\gamma_p(r(p)) = p$, we have

$$v(p) = (p - \cos r(p)\ e_{n+1})|\sin r(p).$$

Now let $\rho : S^{n+1}|\{e_{n+1}, -e_{n+1}\} \to S^n$ be the map which associates to each point p, the point $\gamma_p(\pi/2)$. This map is a kind of Gauss' map, and $\rho(p) = v(p)$. Since

$$\begin{aligned} d\rho_p(w) &= \frac{1}{(\sin r(p))^2} [(w + dr_p(w) \sin r(p)\ e_{n+1}) \sin r(p) \\ &\quad - dr_p(w) \cos r(p)(p - \cos r(p)\ e_{n+1})] \end{aligned}$$

we have that $d\rho_p(w) = 0$, if and only if,

$$\begin{aligned} w &= dr_p(w) \left(\frac{\cos r(p)}{\sin r(p)} (p - \cos r(p)\ e_{n+1}) - \sin r(p)\ e_{n+1} \right) \\ &= dr_p(w)(\cos r(p)\ v(p) - \sin r(p)\ e_{n+1}). \end{aligned}$$

Thus $d\rho_p(w) = 0$, if and only if, w is a scalar multiple of $\gamma'_p(r(p))$, since

$$\gamma'_p(r(p)) = -\sin r(p)\ e_{n+1} + \cos r(p)\ v(p).$$

We will now consider the map $F = \rho \circ x : M^n \to S^n$. Since $e_{n+1} \notin W$, $\gamma_p'(r(p)) \notin dx_p(T_pM)$. Then the map F is a local diffeomorphism. On the other hand, since M^n is compact and S^n is simply connected, F is a diffeomorphism. So x is an embedding, for $F = \rho \circ x$.

In the compact case with constant mean curvature we obtain the following result.

4.2 Theorem. *Let M^n be a connected, compact Riemannian manifold and let $x : M^n \to Q_c^{n+1}$ be an isometric immersion with constant mean curvature H. Then W is nonempty, if and only if, x is umbilic, i.e., $x(M^n)$ is a geodesic sphere of Q_n^{n+1}.*

Proof. Let $p_0 \in W$ and X be the position vector with origin p_0. Since $p_0 \in W$, the support function $g = \langle X, N\rangle$ is nonzero at every point. We can assume that $g > 0$.

From Proposition (2.1), $\Delta g = -\|B\|^2\ g - n\ H\ \theta_c$. By integrating this expression over M^n, and by using Stokes' Theorem, we obtain

$$0 = \int_M \Delta g\ dA = -\int_M (\|B\|^2 g + n\ H\ \theta_c)\ dA.$$

Thus $\int_M \|B\|^2 g\ dA = -n\ H \int_M \theta_c\ dA$. But, from (6),

$$\int_M \|B\|^2 g\ dA = -n\ H \int_M \theta_c\ dA = n\ H^2 \int_M g\ dA.$$

Since $\|B\|^2 \geq nH^2$ and $g > 0$, we have that $\|B\|^2 = nH^2$, which proves that the immersion is umbilic.

4.3 Remark. Alexandrov's Theorem says that if $x : M^n \to Q_c^{n+1}$, $c \leq 0$, is an isometric embedding with constant mean curvature, then $x(M^n)$ is a geodesic sphere. For the case $c > 0$, this result holds if $x(M^n)$ is contained in an hemisphere of Q_c^{n+1}. Therefore, Alexandrov's Theorem, together with Proposition (4.1), gives another proof of Theorem (2.2), with the restriction made above when $c > 0$.

4.4 Remark. Examples of tori in $\mathbb{R}^3$ (see Wente [12]) and nonumbilic immersions $x : S^n \to Q_c^{n+1}$, $c \leq 0$, with constant mean curvature (see Gomes [5], Hsiang [9]) are known. Therefore, in these examples, the set W is empty.

In (L. Barbosa, do Carmo, Eschenburg [2]) the following theorem was proved: Let M^n be a compact Riemannian manifold and $x : M^n \to Q_c^{n+1}$ be an isometric immersion with constant mean curvature. Then x is stable, if and only if, x is umbilic. From this result, we obtain the following Corollary of Theorem (4.2).

4.5 Corollary. *Let M^n be a compact Riemannian manifold, and let $x : M^n \to Q_c^{n+1}$ be an isometric immersion with constant mean curvature. Then W is nonempty, if and only if, x is stable.*

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