STABILITY PROPERTIES OF ROTATIONAL CATENOIDS IN THE HEISENBERG GROUPS

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ABSTRACT. In this paper, we determine the maximally stable, rotationally invariant domains on the catenoids $\mathcal{C}_a$ (minimal surfaces invariant by rotations) in the Heisenberg group. We show that these catenoids have finite Morse index at least 3 and we bound the index from above in terms of the parameter a. We also study the rotationally symmetric stable domains on the higher dimensional catenoids.

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1. INTRODUCTION

Catenoids in Nil(3) are minimal surfaces which are invariant under a one-parameter subgroup of rotations with axis the center of the group. They come in a one-parameter family $\{\mathcal{C}_a, a > 0\}$ of complete minimal surfaces and were first described in [5] and [6] where the authors provide the classification of constant mean curvature surfaces in the Heisenberg group, invariant under certain subgroups of isometries.

In this paper, we study the stability properties of the catenoids $\{\mathcal{C}_a, a > 0\}$ in the Heisenberg groups. More precisely, we determine the rotationally invariant stable domains of the catenoids in Nil$(2n+1)$, $n \geq 1$, with a different behaviour (Lindeloef's property) when $n = 1$ and when $n \geq 2$. We also study the Morse index of the catenoids in Nil(3). As in [3], the proofs rely on a detailed analysis of the Jacobi fields induced from the Killing fields of the ambient Heisenberg space and from the variation of the parameter a.

The paper is organized as follows. In Section 2, we give some preliminary results. We first recall the basic geometry of the Heisenberg group Nil(3) (see [6] for more details). In order to keep our paper self-contained, we derive the differential equation satisfied by the generating curves of the catenoids, using a flux formula. In Section 3, we describe the stable rotationally invariant domains on $\{\mathcal{C}_a\}$ (Theorem 3.1). The proof uses Jacobi fields. We also give some information on the Gauss map of the catenoids $\{\mathcal{C}_a\}$. In Section 4, we prove that the catenoids have finite Morse index at least 3 (Theorem 4.4). The proof uses Jacobi fields, Fourier analysis and an adapted perturbation of the original parametrization of the catenoids. Finally, in Section 5, we

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study the maximally stable, rotationally invariant domains on the higher dimensional catenoids (Theorem 5.1).

In the sequel our functions will often depend on the parameter a. We will occasionally omit a to keep our notations simpler.

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2. Preliminaries

2.1. **The 3-dimensional Heisenberg manifold.** Let Nil(3) denote the 3-dimensional *Heisenberg group*. This is a two-step nilpotent Lie group which can be seen as the subgroup of 3×3 matrices given by

$$\mathrm{Nil}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; (x,y,z) \in \mathbb{R}^3 \right\} \subset GL(3,\mathbb{R}).$$

We denote the corresponding Lie algebra by

$$\mathcal{L}(\mathrm{Nil}(3)) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} ; (x,y,z) \in \mathbb{R}^3 \right\}.$$

Using the exponential map, $\exp : \mathcal{L}(\mathrm{Nil}(3)) \to \mathrm{Nil}(3)$, and the Campbell-Hausdorff formula, we can view Nil(3) as $\mathbb{R}^3$ equiped with the group structure $\star$ given by

$$(x,y,z) \star (x',y',z') = \big(x+x', y+y', z+z' + \frac{1}{2}(xy' - x'y)\big), \tag{1}$$

with neutral element $0 = (0,0,0)$ and inverse $\check{p}$ of $p = (a,b,c)$ given by $\check{p} = (-a,-b,-c)$. The left-multiplication by p in Nil(3), $L_p : q \mapsto p \star q$, has tangent map

$$T_q L_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}b & \frac{1}{2}a & 1 \end{pmatrix} \tag{2}$$

in the canonical coordinates $\{x,y,z\}$ of $\mathbb{R}^3$ (they are often referred to as exponential coordinates). Let $\{\partial_x, \partial_y, \partial_z\}$ be the canonical vector fields in $\mathbb{R}^3$. It follows from the expression (2) that the vector fields

$$\left\{ \begin{array}{rclcl} X(x,y,z) & = & T_0 L_{(x,y,z)}(\partial_x) & = & \partial_x - \frac{y}{2}\,\partial_z, \\ Y(x,y,z) & = & T_0 L_{(x,y,z)}(\partial_y) & = & \partial_y + \frac{x}{2}\,\partial_z, \\ Z(x,y,z) & = & T_0 L_{(x,y,z)}(\partial_z) & = & \partial_z, \end{array} \right. \tag{3}$$

form a basis of left-invariant vector fields in Nil(3).

From now on, we fix the left-invariant metric $\hat{g}$ on Nil(3) to be such that the family $\{X,Y,Z\}$ is an orthonormal frame. In the coordinates $\{x,y,z\}$, this metric is given by

$$\hat{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \big(\mathrm{d}z + \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)\big)^2.$$

The following properties are well-known and can be found for example in [6], Section 1. Equiped with the left-invariant metric $\hat{g}$, the Heisenberg group Nil(3) is a homogeneous Riemannian manifold whose group of isometries has dimension 4. A basis of Killing vector fields on $(\mathrm{Nil}(3), \hat{g})$ is given by

$$\left\{ \begin{array}{rcl} \xi &=& X + yZ, \\ \eta &=& Y - xZ, \\ \zeta &=& Z, \\ \rho &=& yX - xY + \frac{1}{2}(x^2+y^2)Z. \end{array} \right. \tag{4}$$

The first three vector fields ξ, η and ζ correspond to the *left-translations* in Nil(3), while the vector-field ρ corresponds to the one-parameter subgroup of isometries defined by

$$\psi_\theta((x,y,z)) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z), (x,y,z) \in \mathbb{R}^3, \theta \in \mathbb{R}, \tag{5}$$

in the representation $(\mathbb{R}^3, \star)$ of Nil(3). We call them *rotations around the z-axis.* Notice that the z-axis is precisely the center of Nil(3).

2.2. **Surfaces of revolution in** Nil(3)**.** We say that a surface M in Nil(3) is a *surface of revolution* if M is invariant under the action of the one-parameter subgroup $\{\psi_\theta, \theta \in \mathbb{R}\}$ given by (5). We will consider surfaces of revolution whose generating curves are graphs $t \to (f(t), t)$ above the z-axis in the 2-plane $\{x, z\}$, where f is a positive function, and where t varies in some interval $I \subset \mathbb{R}$. They are given by

$$\mathcal{F}(t,\theta) = (f(t)\cos\theta, f(t)\sin\theta, t), \tag{6}$$

for $t \in I \subset \mathbb{R}$ and $\theta \in [0, 2\pi]$.

Catenoids, *i.e.* minimal surfaces of revolution, in Nil(3) are described in [5, 6], using the methods of equivariant differential geometry. They come in a one-parameter family of complete minimal surfaces, $\{\mathcal{C}_a, a > 0\}$. For the sake of completeness and for later purposes, we now derive the differential equation satisfied by the generating curve of a catenoid using a *flux formula* which we now state.

Proposition 2.1. *Let $(M^n, g) \looparrowright (\widehat{M}^{n+1}, \hat{g})$ be an isometric immersion with Riemannian measure μ_g and normalized mean curvature vector $\vec{H}$. Let Ω be a relatively compact smooth domain in M. Let ν_{int} denote the unit normal to $\partial\Omega$ pointing inwards, and σ_g the Riemannian measure on $\partial\Omega$ induced by g. Then, for any Killing vector field $\mathcal{K}$ on $\widehat{M}^{n+1}$, we have*

$$\int_{\partial\Omega} \hat{g}(\mathcal{K}, \nu_{int})\,\mathrm{d}\sigma_g = -n \int_\Omega \hat{g}(\mathcal{K}, \vec{H})\,\mathrm{d}\mu_g. \tag{7}$$

Proof. Let κ be the restriction to M of the 1-form dual to $\mathcal{K}$, i.e. $\kappa = \hat{g}(\mathcal{K}, .)|_M$. A straightforward computation shows that the divergence $\delta_g\kappa$ of the 1-form κ, for the induced metric g on M, is given by

$$\delta_g\kappa = -n\hat{g}(\mathcal{K}, \vec{H}).$$

The proposition follows from the divergence theorem. $\square$

Let $M = \mathcal{F}(I \times [0, 2\pi])$ be a minimal surface of revolution in Nil(3), given by an immersion $\mathcal{F}(t, \theta)$ as in (6), with $t \in I \subset \mathbb{R}$, $\theta \in [0, 2\pi]$. We can make a coherent choice of a unit vector field ν tangent to M and orthogonal to the circles $C_t = \mathcal{F}(\{t\} \times [0, 2\pi])$ in such a way that Proposition 2.1 gives

$$\int_{C_t} \hat{g}(\mathcal{K}, \nu)\, \mathrm{d}\sigma_{C_t} = \int_{C_{t_0}} \hat{g}(\mathcal{K}, \nu)\, \mathrm{d}\sigma_{C_{t_0}}, \tag{8}$$

for all $t_0, t \in I$ and for any Killing vector field $\mathcal{K}$ in Nil(3).

Proposition 2.2. *The generating curve of a minimal surface of revolution in* Nil(3) *of the form (6) satisfies the first order differential equation*

$$f\,(4 + f^2 f_t^2 + 4 f_t^2)^{-1/2} = C \quad \textit{(a constant)}, \tag{9}$$

and the second order differential equation

$$f(4 + f^2)\, f_{tt} = 4\,(1 + f_t^2), \tag{10}$$

where f_t and f_{tt} denote respectively the first and second derivatives of the function f with respect to the variable t.

Proof. According to [6] Theorem 3, we already know that minimal surfaces of revolution do exist in Nil(3). Equation (9) is established by applying Proposition 2.1 with the Killing field $\mathcal{K} = Z$. The constant C can then be interpreted in terms of a flux. The vectors $\mathcal{F}_t$ and $\mathcal{F}_\theta$ are tangent to the surface. Using (3), they can be expressed in the orthonormal frame $\{X, Y, Z\}$ as

$$\begin{cases} \mathcal{F}_t &= f_t \cos\theta\, X + f_t \sin\theta\, Y + Z, \\ \mathcal{F}_\theta &= -f \sin\theta\, X + f \cos\theta\, Y - \frac{1}{2} f^2\, Z. \end{cases} \tag{11}$$

The Riemannian measure σ_{C_t} is given by

$$\mathrm{d}\sigma_{C_t} = \sqrt{\hat{g}(\mathcal{F}_\theta, \mathcal{F}_\theta)}\, \mathrm{d}\theta = f \sqrt{1 + \frac{1}{4} f^2}\, \mathrm{d}\theta.$$

Up to sign, the vector ν is characterized by the facts that it is unitary, tangent to the surface – hence a linear combination of $\mathcal{F}_t$ and $\mathcal{F}_\theta$ – and orthogonal to $\mathcal{F}_\theta$. Consider the vector $n = \mathcal{F}_t + \alpha \mathcal{F}_\theta$ with α such that $\hat{g}(n, \mathcal{F}_\theta) = 0$. We choose $\nu = \hat{g}(n, n)^{-1/2} n$. The expression $\hat{g}(Z, \nu)$ which appears in (8) is the Z-component of ν. A straightforward computation gives that $\alpha = 2(4 + f^2)^{-1}$, $\hat{g}(n, Z) = 4(4 + f^2)^{-1}$ and $\hat{g}(n, n) = f_t^2 + 4(4 + f^2)^{-1}$. It follows that

$$\hat{g}(Z, \nu) = 4(4 + f^2)^{-1} (f_t^2 + \frac{4}{4 + f^2})^{-1/2}.$$

Using (8),we obtain that the quantity

$$f(t)\,[4 + f^2(t)\, f_t^2(t) + 4 f_t^2(t)]^{-1/2}$$

is independent of t (a flux). Equation (9) follows. Taking the derivative of (9) and using the fact that $f_t \not\equiv 0$ (see [6]), we obtain Equation (10). □

Remark. The above equations can also be derived directly from [6] (using the computations in the proof of their Theorem 3) or by minimizing the area of a rotational domain, in the spirit of the calculus of variations.

2.3. **Qualitative analysis of Equation (10).** Given $a > 0$, consider the *Cauchy problem*,

$$
\left\{
\begin{aligned}
f(f^2+4)f_{tt} &= 4(1+f_t^2),\\
f(0) &= a,\\
f_t(0) &= 0,
\end{aligned}
\right.
\tag{12}
$$

where the subscript t denotes derivative with respect to t. Recall that this differential equation admits a first integral and, more precisely, that

$$
\frac{(f^2+4)(1+f_t^2)}{f^2} = \frac{a^2+4}{a^2}.
\tag{13}
$$

An easy analysis shows that (12) admits a maximal solution $f(a,t)$ which is an even function of t. Furthermore, the function

$$f(a,\cdot) : [0, A_a) \to [a, \infty)$$

is an increasing function and we can introduce its inverse function

$$\phi(a,\cdot) : [a,\infty) \to [0, A_a).$$

Using (13), we infer that ϕ is given by the integral

$$
\phi(a,\tau) = \frac{a}{2}\int_1^{\tau/a} \sqrt{\frac{a^2v^2+4}{v^2-1}}\,\mathrm{d}v.
\tag{14}
$$

It follows that

$$
\phi(a,\tau) \sim \frac{a}{2}\tau, \quad \text{when} \quad \tau \to \infty.
\tag{15}
$$

Finally, we conclude that the Cauchy problem (12) admits a global solution $f(a,\cdot) : \mathbb{R} \to [a,\infty)$ which satisfies

$$
\left\{
\begin{aligned}
& f(a,t) = f(a,-t),\\
& f(a,t) \sim \tfrac{2}{a}|t|, \quad \text{and}\\
& f_t(a,t) \sim \tfrac{2}{a}\operatorname{sgn}(t), \quad \text{when} \quad |t| \to \infty.
\end{aligned}
\right.
\tag{16}
$$

2.4. **The Jacobi operator of minimal surfaces.** In this section, we recall some classical definitions and facts about the Jacobi operator of minimal surfaces. Let $M^2 \looparrowright \widehat{M}^3$ be an orientable minimal surface immersed into an oriented Riemannian manifold $(\widehat{M}, \hat{g})$. Let N_M be a unit normal field along M, A_M the second fundamental form of the immersion with respect to the normal N_M, and let $\widehat{\mathrm{Ric}}$ be the Ricci curvature of $\widehat{M}$. The second variation of the volume functional gives rise to the *Jacobi operator* J_M of M (see [7])

$$
J_M := -\Delta_M - (|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)),
\tag{17}
$$

where Δ_M is the non-positive Laplacian on M for the induced metric.

Given a relatively compact regular domain Ω on the surface M, we let $\mathrm{Ind}(\Omega)$ denote the number of negative eigenvalues of J_M for the Dirichlet problem in Ω. The *Morse index* of M is defined to be the supremum

$$\mathrm{Ind}(M) := \sup\{\mathrm{Ind}(\Omega); \Omega \Subset M\} \le \infty,$$

taken over all relatively compact regular domains. Let $\lambda_1(\Omega)$ be the least eigenvalue of the operator J_M with the Dirichlet boundary conditions in Ω.

We call a relatively compact regular domain Ω *stable* if $\lambda_1(\Omega) > 0$, *unstable* if $\lambda_1(\Omega) < 0$, and *stable-unstable* if $\lambda_1(\Omega) = 0$. More generally, we say that a domain Ω is stable if any relatively compact subdomain is stable. We collect classical results we will need later on in the following proposition.

Proposition 2.3. *Given a minimal immersion $M^2 \looparrowright \widehat{M}^3$, the following properties hold.*

1. *Let Ω be a stable-unstable relatively compact domain. Then, any smaller domain is stable while any larger domain is unstable.*
2. *We refer to the solutions of the equation $J_M(u) = 0$ as* Jacobi fields *on M. Let $X_a : M^2 \looparrowright (\widehat{M}^3, \hat{g})$ be a one-parameter family of oriented minimal immersions, with variation field $V_a = \frac{\partial X_a}{\partial a}$ and with unit normal N_a. Then, the function $\hat{g}(V_a, N_a)$ is a Jacobi field on M.*
3. *Let Ω be a relatively compact domain on a minimal submanifold M. If there exists a positive function u on Ω such that $J_M(u) \geq 0$, then Ω is stable or stable-unstable.*

Proof. *Assertion 1* follows from the min-max characterization of eigenvalues and the maximum principle. *Assertion 2* appears in [1] (Theorem 2.7 and its proof) in a more general framework. *Assertion 3* is proved in [4], see the proof of Theorem 1. $\square$

3. Stable domains of revolution on the catenoids

We consider a catenoid $\mathcal{C}$ given by the map,

$$\mathcal{F} : \mathbb{R} \times [0, 2\pi] \to \mathcal{C} \looparrowright \mathrm{Nil}(3),$$

$$\mathcal{F}(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t),$$

where f is a global solution of (10).

It follows from (11) that the coefficients of the first fundamental form induced by $\mathcal{F}$ and the square root D of its determinant are given by

$$\left\{\begin{aligned} E &= \; 1 + f_t^2, \\ F &= \; -\tfrac{1}{2} f^2, \\ G &= \; f^2(1 + \tfrac{1}{4} f^2), \text{ and} \\ D &= \; f\big(1 + f_t^2 + \tfrac{1}{4} f^2 f_t^2\big)^{1/2}. \end{aligned}\right. \tag{18}$$

Let N be a unit normal field to $\mathcal{F}$. Writing $N = \alpha X + \beta Y + \gamma Z$, we find that

$$\left\{\begin{aligned} \alpha &= W(-\cos\theta - \tfrac{1}{2} f f_t \sin\theta), \\ \beta &= W(-\sin\theta + \tfrac{1}{2} f f_t \cos\theta), \\ \gamma &= W f_t, \quad \text{where} \\ W &= (1 + f_t^2 + \tfrac{1}{4} f^2 f_t^2)^{-1/2}. \end{aligned}\right. \tag{19}$$

3.1. **Jacobi fields coming from ambient Killing fields.** Since $\{\xi, \eta, \zeta, \rho\}$ is a basis of Killing vector fields, it follows from Proposition 2.3(2) that the functions

$$\begin{cases} v_\xi = & \hat{g}(\xi, N) = & W(-\cos\theta + \frac{1}{2} f f_t \sin\theta), \\ v_\eta = & \hat{g}(\eta, N) = & W(-\sin\theta - \frac{1}{2} f f_t \cos\theta), \\ v_\zeta = & \hat{g}(\zeta, N) = & W f_t, \end{cases} \tag{20}$$

are Jacobi fields on the surface $\mathcal{F}$ (note that $v_\rho = \hat{g}(\rho, N) = 0$).

Remark. The Jacobi fields v_ξ, v_η and v_ζ are linearly independent.

3.2. **A Jacobi field coming from the variation of the family.** We now consider the 1-parameter family of catenoids $\{\mathcal{C}_a, a > 0\}$, generated from the family of maps

$$\mathcal{F}(a, t, \theta) = (f(a, t) \cos\theta, f(a, t) \sin\theta, t), \quad a > 0, \tag{21}$$

where $f(a, \cdot)$ is the unique global solution of the Cauchy problem (12). The variational field of this family is given by

$$\mathcal{F}_a(a, t, \theta) = f_a(a, t) \cos\theta \, X + f_a(a, t) \sin\theta \, Y. \tag{22}$$

Here $f_a(a, t) := \frac{\partial f}{\partial a}(a, t)$. By Proposition 2.3(2), this yields another Jacobi field on $\mathcal{C}_a$, namely, $e(a, \cdot) = -\hat{g}(\mathcal{F}_a, N)$. More precisely,

$$e(a, t) = \big(W f_a\big)(a, t), \tag{23}$$

where the function W is given by the last line in (19). We note that $e(a, \cdot)$ does not depend on θ and is an even function of t. Futhermore, since $f(a, 0) = a$, $\forall a > 0$, we have $e(a, 0) = 1$, $\forall a > 0$.

The rotationally invariant stable domains of the catenoids $\mathcal{C}_a$ are described in the following theorem.

Theorem 3.1. *Let $\mathcal{C}_a$ be a catenoid in* Nil(3). *Then*

(1) *The upper (resp. the lower) half catenoid $\mathcal{C}_{a,+} = \mathcal{C}_a \cap \{z > 0\}$ (resp. $\mathcal{C}_{a,-} = \mathcal{C}_a \cap \{z < 0\}$) is stable.*
(2) *The function $e(a, \cdot)$ is even and has exactly one zero, $z(a)$ on $(0, \infty)$. The domain $\mathcal{F}(a, [-z(a), z(a)], [0, 2\pi])$ is a stable-unstable domain in $\mathcal{C}_a$.*
(3) *Given any $t_1 > 0$, there exists some $t_2 > 0$ such that the domain $\mathcal{D}_a(-t_1, t_2) = \mathcal{F}(a, [-t_1, t_2], [0, 2\pi])$ is stable-unstable. This implies in particular that both $\mathcal{C}_{a,+}$ and $\mathcal{C}_{a,-}$ are maximal stable rotationally invariant domains (*i.e. *in some sense, stable-unstable).*

Proof. *Assertion 1.* It follows from Section 2.3 that the Jacobi field v_ζ is positive on $(0, +\infty)$ and negative on $(-\infty, 0)$. The assertion follows from Proposition 2.3(3).

Assertion 2. We already know that $e(a, \cdot)$ is an even function of t and that $e(a, 0) = 1$ for all $a > 0$. *Claim 1.* The function $e(a, \cdot)$ has at most one zero in $(0, +\infty)$. If not, $e(a, \cdot)$ would have two consecutive positive zeroes, $0 < z_1(a) < z_2(a)$ and the domain $\mathcal{F}(a, [z_1(a), z_2(a)], [0, 2\pi])$ would be stable-unstable. According to Proposition 2.3(1), this would contradict the stability of $\mathcal{C}_{a,+}$ in Assertion 1. *Claim 2.* The function $e(a,\cdot)$ has at least one zero in $(0,+\infty)$. Indeed, $e(a,\cdot)$ has the sign of $f_a(a,t)$. Using the function ϕ defined by (14), we find that

$$\phi_a(a, f(a,t)) + f_a(a,t)\, \phi_\tau(a, f(a,t)) \equiv 0$$

for all $a, t > 0$. Since ϕ_τ is positive, it suffices to look at the sign of ϕ_a. We find that

$$(*) \qquad \phi_a(a,\tau) = \int_1^{\tau/a} \frac{a^2v^2+2}{\sqrt{(a^2v^2+4)(v^2-1)}}\, \mathrm{d}v - \frac{\tau}{2}\sqrt{\frac{\tau^2+4}{\tau^2-a^2}}$$

and we easily conclude that $\phi_a(a,\tau)$ is positive when τ is large enough. It follows that $e(a,t)$ is negative for t large enough so that it must vanish at least once in $(0,+\infty)$.

Assertion 3. Fix some $t_1 > 0$ and consider the function

$$w(a,t_1,t) = v(a,t_1)e(a,t) + e(a,t_1)v(a,t),$$

where we have written $v(a,t)$ instead of $v_\zeta(a,t)$ for short. This is a Jacobi field on $\mathcal{C}_a$, which vanishes at $t = -t_1$. Note that $w(a,t_1,0) = v(a,t_1) > 0$ because $e(a,0) = 1$ and $v(a,t) > 0$ for any $t > 0$. As in the proof of Assertion 2, Claim 1, we see that $w(a,t_1,\cdot)$ can vanish at most once in $(-\infty,0)$ and $(0,\infty)$. It follows that $w(a,t_1,\cdot)$ has exactly one zero in $(-\infty,0)$ (namely $-t_1$) and that it vanishes in $(0,\infty)$ if and only if it takes some negative value near infinity. Recall that

$$(a) \qquad v(a,t) = \frac{f_t}{\sqrt{1+f_t^2+\frac{1}{4}f_t^2f^2}}(a,t).$$

As in the proof of Assertion 2, Claim 2, we use the functional equation $\phi(a,f(a,t)) \equiv t$ for all $t > 0$ and the relation $\phi_\tau(a,f(a,t))\, f_t(a,t) \equiv 1$ for all $t > 0$. Plugging this relation into (a), we find that

$$(b) \qquad v(a,t) = \tilde{v}(a,f(a,t)), \quad \forall t > 0,$$

where $\tilde{v}(a,\tau) = \left(1 + \frac{\tau^2}{4} + \phi_\tau^2(a,\tau)\right)^{-1/2}$. Similar computations yield the relation

$$(c) \qquad e(a,t) = -\phi_a(a,f(a,t))v(a,t) = \tilde{e}(a,f(a,t)), \quad \forall t > 0,$$

where $\tilde{e}(a,\tau) = -\phi_a(a,\tau)\tilde{v}(a,\tau)$. Define

$$(d) \qquad \tilde{w}(a,t_1,\tau) = v(a,t_1)\tilde{e}(a,\tau) + e(a,t_1)\tilde{v}(a,\tau),$$

so that $w(a,t_1,t) = \tilde{w}(a,t_1,f(a,t))$. Then,

$$(e) \qquad \tilde{w}(a,t_1,\tau) = -\tilde{v}(a,\tau)v(a,t_1)\big(\phi_a(a,\tau) + \phi_a(a,\tau_1)\big)$$

where $\tau_1 := f(a,t_1)$. Using $(*)$, we see that w is negative near infinity, for any $a, t_1 > 0$. This proves the existence of a positive t_2 such that the domain $\mathcal{D}_a(-t_1,t_2)$ is stable-unstable. The last assertion follows immediately. $\square$

Remark. Using (19) and Section 2.3, we can see that the Gauss map of the catenoid $\mathcal{C}_a$ covers a closed symmetric strip about the equator of the unit sphere in the Lie algebra $\mathcal{L}(\mathrm{Nil}(3))$. This strip, whose width depends on a, is strictly contained in the sphere minus the south and north poles. Each point of the open strip is covered exactly twice, except the points of the equator which are covered once (look at the variations of the Z-component γ of the vector N).

4. The index of the catenoids $\mathcal{C}_a$ in Nil(3)

In this section, we study the Morse index of the catenoids $\mathcal{C}_a$. It turns out that the representation $\mathcal{F}$ given by (6), with the function f satisfying (10), is not well-adapted to Fourier analysis on $\mathcal{C}_a$ because the vectors $\mathcal{F}_t$ and $\mathcal{F}_\theta$ are not orthogonal. To avoid this problem, we introduce a perturbed representation,

$$\widetilde{\mathcal{F}}(t,\theta) := \mathcal{F}(t,\theta+\varphi(t)) = \big(f(t)\cos(\theta+\varphi(t)), f(t)\sin(\theta+\varphi(t)), t\big). \tag{24}$$

The tangent vectors are given by

$$\begin{cases} \widetilde{\mathcal{F}}_t(t,\theta) = & \mathcal{F}_t(t,\theta+\varphi(t)) + \varphi_t(t)\mathcal{F}_\theta(t,\theta+\varphi(t)),\\ \widetilde{\mathcal{F}}_\theta(t,\theta) = & \mathcal{F}_\theta(t,\theta+\varphi(t)). \end{cases}$$

It follows that the representation $\widetilde{\mathcal{F}}$ is orthogonal – *i.e.* the vectors $\widetilde{\mathcal{F}}_t$ and $\widetilde{\mathcal{F}}_\theta$ are orthogonal – if and only if the function φ satisfies the differential equation

$$\varphi_t = \frac{2}{4+f^2}. \tag{25}$$

From now on, we choose φ to be the solution of (25) such that $\varphi(0)=0$.

Note that in the above expressions, we have omitted the dependence on the parameter a. The unit normal vector to $\mathcal{C}_a$ at the point $\widetilde{\mathcal{F}}(t,\theta)$ is $\widetilde{N}(t,\theta) = N(t,\theta+\varphi(t))$. In the representation $\widetilde{\mathcal{F}}$, the Riemannian metric induced by the immersion $\mathcal{C}_a \looparrowright \mathrm{Nil}(3)$ is of the form $D^2G^{-1}\mathrm{d}t^2 + G\mathrm{d}\theta^2$, with the functions D, G as in (18). It follows that the Laplacian on $\mathcal{C}_a$ is given, in the representation $\widetilde{\mathcal{F}}$, by the expression

$$\widetilde{\Delta} = \frac{1}{D}\,\partial_t\Big(\frac{G}{D}\partial_t\Big) + \frac{1}{G}\,\partial^2_{\theta\theta}. \tag{26}$$

We introduce the operator

$$\widetilde{L} = -\frac{1}{D}\,\partial_t\Big(\frac{G}{D}\partial_t\Big), \tag{27}$$

and the function

$$\widetilde{V} = \big(\widehat{\mathrm{Ric}}(\widetilde{N}) + |\widetilde{A}|^2\big), \tag{28}$$

which only depend on the variable t (and the parameter a). The Jacobi operator (17) of the immersion $\mathcal{C}_a \looparrowright \mathrm{Nil}(3)$ is given by the expression

$$\widetilde{J} = \widetilde{L} - \widetilde{V} - \frac{1}{G}\,\partial^2_{\theta\theta}. \tag{29}$$

We have the following lemma.

Lemma 4.1. *With the above notations, the function $\widetilde{V}$ on the catenoid $\mathcal{C}_a$ is given by,*

$$\widetilde{V} = \frac{2a^2}{f^2}\Big(\frac{1}{f^2} + \frac{1+f_t^2}{4+f^2}\Big) = \frac{2a^2}{f^2}\Big(\frac{1}{f^2} + \frac{a^2+4}{a^2}\,\frac{f^2}{(4+f^2)^2}\Big). \tag{30}$$

Proof. For the catenoid $\mathcal{C}_a$, the function f satisfies the differential equations (13) and (10) and we have $W = \frac{a}{f}$, where the function W is defined in (19). The Z-component γ of the unit normal $\widetilde{N}$ is a Jacobi field, hence $\widetilde{L}(\gamma) = \widetilde{V}\gamma$. Using (13) and (10) again, we can compute $\widetilde{L}(\gamma)$ and derive the formulas for $\widetilde{V}$ on the catenoid $\mathcal{C}_a$. $\square$

Let $\widetilde{v}_\xi$ and $\widetilde{v}_\eta$ be the expressions of the Jacobi fields associated with the Killing fields ξ and η. It follows from (20) that

$$\widetilde{v}_\xi(t,\theta) = \hat{g}\Big(\xi(\widetilde{\mathcal{F}}(t,\theta)), \widetilde{N}(t,\theta)\Big) = W\Big(-\cos(\theta+\varphi) + \frac{1}{2} f f_t \sin(\theta+\varphi)\Big),$$

and similarly for $\widetilde{v}_\eta$ (we have omitted the dependence on a). We introduce the smooth function $\psi(a,t)$ such that

$$\begin{cases} \cos\psi &= (1+\frac{1}{4}f^2 f_t^2)^{-1/2},\\ \sin\psi &= \frac{1}{2} f f_t (1+\frac{1}{4}f^2 f_t^2)^{-1/2},\\ \psi(a,0) &= 0. \end{cases} \tag{31}$$

It follows immediately that

$$\begin{cases} \widetilde{v}_\xi(a,t,\theta) &= -W_1(a,t)\cos\big(\theta+\varphi(a,t)+\psi(a,t)\big),\\ \widetilde{v}_\eta(a,t,\theta) &= W_1(a,t)\sin\big(\theta+\varphi(a,t)+\psi(a,t)\big), \quad \text{where}\\ W_1 &= W(1+\frac{1}{4}f^2 f_t^2)^{1/2}. \end{cases} \tag{32}$$

With the above notations, we have the following lemma.

Lemma 4.2. *Let $\omega := \varphi + \psi$, a function of the variable t and the parameter a. Then,*

(1) *The functions*

$$\begin{cases} w_1(a,t,\theta) := W_1(a,t)\cos(\omega(a,t))\cos\theta,\\ w_2(a,t,\theta) := W_1(a,t)\cos(\omega(a,t))\sin\theta,\\ w_3(a,t,\theta) := W_1(a,t)\sin(\omega(a,t))\cos\theta,\\ w_4(a,t,\theta) := W_1(a,t)\sin(\omega(a,t))\sin\theta, \end{cases} \tag{33}$$

are Jacobi fields on $\mathcal{C}_a$, $\widetilde{J}(w_i) = 0$, for $1 \le i \le 4$.

(2) *The function $\omega(a,\cdot)$ is an odd function of t, satisfying $\omega(a,0) = 0$ and $\omega_t = 4f^2(f^4+4a^2)^{-1}$.*

(3) *Let $\Omega(a) := \lim_{t\to+\infty}\omega(a,t)$. Then*

$$\Omega(a) = 2a\int_a^\infty \frac{u^2\sqrt{u^2+4}}{(u^4+4a^2)\sqrt{u^2-a^2}}\,\mathrm{d}u. \tag{34}$$

(4) *For all $a > 0$, we have $\frac{\pi}{2} < \Omega(a) < \pi$ and the lower and upper bounds are achieved as limits when a tends respectively to zero and infinity.*

Proof. *Assertion 1* follows from the equalities $\widetilde{v}_\xi = -w_1 + w_4$ and $\widetilde{v}_\eta = w_2 + w_3$, and the fact that the operator $\widetilde{J}$ separates variables. *Assertion 2.* The computation of ω_t is straightforward. To prove *Assertion 3*, we use the fact that f_t is positive for positive t and can be computed from (13), namely,

$$f_t = \frac{2\sqrt{f^2-a^2}}{a\sqrt{f^2+4}}.$$

We write

$$\omega_t = \frac{2af^2\sqrt{f^2+4}}{(f^4+4a^2)\sqrt{f^2-a^2}}\, f_t$$

for $t > 0$, and we compute the integral $\int_0^t \omega_\tau \, \mathrm{d}\tau$ by making the change of variables $u = f(t)$. *Assertion 4.* Assume by contradiction that $\Omega(a_0) \geq \pi$ for some a_0. There would then exist a value t_0 such that $\omega(a_0, t_0) = \pi$. The function w_3, see (33), would then vanish on the circles $\widetilde{\mathcal{F}}(\{0\}, [0, 2\pi])$ and $\widetilde{\mathcal{F}}(\{t_0\}, [0, 2\pi])$. Because this function is a Jacobi field, this would contradict Assertion (1) in Theorem 3.1. The fact that $\frac{\pi}{2} < \Omega(a)$ follows by estimating the integral, [8]. □

Lemma 4.3. *Consider the operator $\widetilde{L_k} := \widetilde{L} + \frac{k^2}{G} - \widetilde{V}$ in $L^2([-r, r], D\,\mathrm{d}t)$, with Dirichlet boundary conditions. Then,*

1. *The operator $\widetilde{L_k}$ has at most one negative eigenvalue (with multiplicity one).*
2. *For all $k \geq \sqrt{a^2+2}$ and $r > 0$, the operator $\widetilde{L_k}$ is positive in $L^2([-r, r], D\,\mathrm{d}t)$.*

Proof. *Assertion 1.* Recall that the eigenvalues of a Sturm-Liouville problem with Dirichlet boundary conditions are all simple. If $\widetilde{L_k}$ had at least two negative eigenvalues, we would have an eigenfunction v of $-\widetilde{L_k}$ associated with a negative eigenvalue and having one zero in $(-r, r)$. The function $v\cos(k\theta)$ would be an eigenfunction of the Jacobi operator J with negative eigenvalue, vanishing on the boundary of an annulus contained in $\mathcal{C}_{a,\pm}$, contradicting Assertion (1) in Theorem 3.1. *Assertion 2.* Using Lemma 4.1 and (18), we see that $G\widetilde{V} \leq a^2 + 2$ and the second assertion follows from the positivity of the operator $\widetilde{L}$ in $L^2([-r, r], D\,\mathrm{d}t)$. □

Theorem 4.4. *Consider the catenoids $\mathcal{C}_a$ in* Nil(3)*. For all $a > 0$, the catenoid $\mathcal{C}_a$ has finite Morse index, at least equal to* 3 *and at most* $1 + 2[\sqrt{a^2+2}]$*, where $[x]$ is the integer part of x. In particular the index is* 3 *for a close to zero.*

Proof. The fact that the index is at least 1 follows from Theorem 3.1. The fact that the index is finite follows from the second assertion in Lemma 4.3. More precisely Fourier analysis and Lemma 4.3(1) show that the Morse index is 1 plus twice the number of positive k such that the operator $\widetilde{L}_k$ has a negative eigenvalue. This latter number can be bounded from above using Lemma 4.3(2). Since $\Omega(a) \geq \pi/2$ and $\omega(0) = 0$, we can choose r such that $\omega(a, r) = \pi/2$. Using the Jacobi field w_3, we see that the operator $\widetilde{L_1}$ has 0 as eigenvalue in $[-r, r]$. It follows that it has a negative eigenvalues in $[-r', r']$ for r' slightly larger than r, with corresponding eigenfunction v_1. On the other-hand, we know that the operator $\widetilde{L}$ has a negative eigenvalue in $[-r', r']$ for r' large enough with eigenfunction v_0. As a consequence, the functions v_0, $v_1\cos\theta$ and $v_1\sin\theta$ are eigenfunctions with negative eigenvalues for the Jacobi operator in some $\widetilde{\mathcal{F}}([-r', r'] \times [0, 2\pi])$. It follows that the index is at least 3. □

Remarks.

(1) Given $a > 0$, there is a simple numerical analysis criterion to decide whether the operator $\widetilde{L}_k$ has a negative eigenvalue in the interval $[-r, r]$ (with Dirichlet boundary conditions). Let u_k be the solution of the Cauchy problem $\widetilde{L}_k(u) = 0$, $u(0) = 1$ and $u_t(0) = 0$. If u_k has a zero in the interval $(0, r)$, then $\widetilde{L}_k$ has a negative eigenvalue in $[-r, r]$; if u_k does not vanish in the interval $(0, r)$, then $\widetilde{L}_k(u) \geq 0$ in $[-r, r]$.

(2) Using the fact that the metric $\hat{g}$ on Nil(3) is left-invariant, one can easily express the associated Levi-Civita connexion and curvature tensors on the orthonormal basis $\{X, Y, Z\}$ of left-invariant vector fields. In particular, given a unit vector $N = \alpha X + \beta Y + \gamma Z$, we find the following formula for the Ricci curvature,

$$\widehat{\mathrm{Ric}}(N, N) = -\frac{1}{2} + \gamma^2.$$

(3) Using the preceding remark, we can write the Jacobi operator on an orientable minimal surface in Nil(3) as

$$J = -\Delta + \frac{1}{2} - \gamma^2 - |A|^2,$$

where γ is the Z-component of the unit normal to the surface. Using the fact that the scalar curvature of Nil(3) is $-\frac{1}{4}$, we also have the formula

$$J = -\Delta + \frac{1}{4} + K_M - \frac{1}{2}|A|^2,$$

where K_M is the Gauss curvature of the surface M.

(4) Using Lemma 4.1 and the first remark, we deduce the following expression for the second fundamental form of the catenoid $\mathcal{C}_a$ in Nil(3),

$$|A|^2 = \frac{1}{2} - \frac{4}{f^2} + \frac{4(a^2+4)}{f^2(f^2+4)} + \frac{2(a^2+4)}{(f^2+4)^2}.$$

This shows that the second fundamental form tends to $\frac{1}{2}$ uniformly at infinity. This is in contrast with the situation in $\mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^3$.

5. Catenoids in higher dimensions

In this section, we study the rotationally symmetric stable domains on the higher dimensional catenoids. Let Nil$(2n+1)$ be the $(2n+1)$-dimensional Heisenberg group. As in Section 2, we use the exponential coordinates and choose the left-invariant metric $\hat{g}$ such that the left-invariant vector fields $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z\}$ form an orthonormal basis, where

$$\begin{cases} X_i(x,y,z) & = \partial_{x_i} - \frac{1}{2} y_i \partial_z, \ 1 \leq i \leq n, \\ Y_i(x,y,z) & = \partial_{y_i} + \frac{1}{2} x_i \partial_z, \ 1 \leq i \leq n, \\ Z(x,y,z) & = \partial_z. \end{cases}$$

We look for hypersurfaces of revolution of the form

$$\mathcal{F} : \begin{cases} \mathbb{R} \times S^{2n-1} \to \mathrm{Nil}(2n+1), \\ (t, \theta) \mapsto \mathcal{F}(t, \theta) = (f(t)\theta, t), \end{cases} \tag{35}$$

where f is a positive function of t. If follows from [5, 6] that such an hypersurface is minimal if and only if f satisfies the second order differential equation,

$$f(4 + f^2) f_{tt} = 4(2n-1)(1 + f_t^2) + (2n-2) f^2 f_t^2. \tag{36}$$

As in Section 2.3, one can show that for $a > 0$, there is a unique maximal solution $f(a, t)$ such that $f(a, 0) = a$ and $f_t(a, 0) = 0$. This is an even function of t defined on the interval $(-T(a), T(a))$, where $T(a)$ is finite when $n \geq 2$. As in dimension 3 ($n = 1$), the above differential equation admits a first integral,

$$f^{2n-1} \left(1 + f_t^2 + f^2 f_t^2\right)^{-1/2} \equiv a^{2n-1}. \tag{37}$$

As in (19), we let $W := \left(1 + f_t^2 + f^2 f_t^2\right)^{-1/2}$. We also use the following notations,

$$\begin{cases} \mathcal{C}_a & = \mathcal{F}\Big(a, (-T(a), T(a)), S^{2n-1}\Big), \\ \mathcal{C}_{a,+} & = \mathcal{F}\Big(a, (0, T(a)), S^{2n-1}\Big), \\ \mathcal{C}_{a,-} & = \mathcal{F}\Big(a, (-T(a), 0), S^{2n-1}\Big), \\ \mathcal{D}_a(r, s) & = \mathcal{F}\Big(a, (r, s), S^{2n-1}\Big). \end{cases} \tag{38}$$

We can now state the following result.

Theorem 5.1. *Assume that $n \geq 2$ and $a > 0$.*

(1) *The half-catenoids $\mathcal{C}_{a,\pm}$ are stable.*
(2) *There exists some $z(a) > 0$ such that the domain $\mathcal{D}_a(-z(a), z(a))$ is stable-unstable. In particular, the catenoid $\mathcal{C}_a$ has index at least 1.*
(3) *There exists some $\ell(a) > 0$ such that the domain $\mathcal{D}_a(-\ell(a), T(a))$ is stable.*
(4) *For any $r > \ell(a)$, there exists some $s > 0$ such that the domain $\mathcal{D}_a(-r, s)$ is stable-unstable.*

Proof. The proof relies on the expressions of two explicit Jacobi fields on $\mathcal{C}_a$, namely the Jacobi fields $v(a, t) = \hat{g}(N, Z)$, and $e(a, t) = -\hat{g}(\mathcal{F}_a, N)$, where N is a unit normal to $\mathcal{C}_a$, and $\mathcal{F}_a$ is the variation field along $\mathcal{F}$ when the parameter a varies. As in dimension 2, we have $v(a, t) = W(a, t) f_t(a, t)$ and Assertion (1) follows immediately from the fact that $f_t(a, t) > 0$ for $t > 0$.

To prove the other Assertions, notice that $e(a, t)$ is an even function of t which can be studied using the inverse function $\phi(a, \tau)$ of the function $f(a, \cdot) : [0, \infty) \to [a, T(a))$. It turns out that

$$\phi(a, \tau) = \frac{a^{2n-1}}{2} \int_a^\tau \sqrt{\frac{u^2 + 4}{u^{4n-2} - a^{4n-2}}}\, \mathrm{d}u. \tag{39}$$

This formula shows that $\phi(a, \tau)$ has a finite limit $T(a)$ when τ tends to infinity and that its derivative $\phi_a(a, \tau)$ has a positive finite limit when τ tends to infinity. We now use the same method as in the proof of Theorem 3.1. Assertion 2, follows from the fact that $e(a,0)=1$ and that $e(a,t)$ takes negative values near infinity. For the proofs of Assertions (3) and (4), we use the fact that in higher dimensions ($n \geq 2$), both $\phi(a,\tau)$ and $\phi_a(a,\tau)$ have finite limits at infinity, so that the higher dimensional case differs from the case in which $n=1$. □

Remark. Theorem 3.1(3) tells us that the half-catenoids $\mathcal{C}_{a,\pm}$ in Nil(3) are stable-unstable, *i.e.* that they satisfy the *Lindeloef's property* as defined in [2, 3]. Theorem 5.1(3) and (4) tell us that catenoids in Nil$(2n+1)$, $n \geq 2$, do not satisfy Lindeloef's property. As for catenoids in $\mathbb{R}^{n+2}$ and $\mathbb{H}^n \times \mathbb{R}$, $n \geq 2$, this is related to the fact that these catenoids have finite height.

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